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THE

ELEMENTS OF ALGEBRA.

BY

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PREFACE.

This book was begun with the hope of preparing something for beginners in Algebra which would give them, in a narrow compass, a philosophical and, therefore, a thorough start in their analytical studies. By adopting a somewhat new arrangement of the subject, it was found that the matter the author had in mind could be presented in such a small space, that he determined to extend the scope of the work far enough to give all that is really necessary to enable the student to prosecute with profit the higher mathematics. This he did the more readily, since he is of the opinion that the further course in Algebra presents rather more difficulties than any other branch of mathematics, and that, consequently, the student at the beginning of his analytical course can derive little real benefit from his efforts to master it. In truth, so far as his information goes, it is generally either omitted altogether, or so little appreciated by the young student, as to be almost lost labor. It is, withal, so important, that it ought to be insisted upon, and so should be taken up at a later stage of the student's course, and then presented in connection with the general philosophy of analytical investigations. Accordingly, if this little work meets with approval, the author will be encouraged to attempt the preparation of a supplementary work to meet this end.

It is not necessary to say much with regard to this essay. It will be found in great part novel in its treatment of the subject. The discussions are considerably extended, and at all points the rationale of operations fully brought out. An effort has been made to preserve the continuity of the subject, so as to present a harmonious whole. The author hopes that it may, in some measure, prove the means of stimulating the reasoning powers of such lads as may chance to use it, and that the mere mechanism of operations will not content them.

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THE ELEMENTS OF ALGEBRA.

SECTION I.

DEFINITIONS AND EXPLANATIONS.

1. *Algebra* is a branch of Mathematics in which letters are commonly used to represent quantities, while the operations to be performed upon them are merely indicated.

2. The leading letters of the alphabet (a, b, c, etc.) are used to denote quantities whose values are given, or may be assumed at pleasure. They are called *known* or *arbitrary quantities.*

Any numeral, as 5 or 25, is made up of a fixed number of units, and admits of no change whatever in this regard : 5 cannot represent 25, or any other number than itself. But if we say that the letter a shall stand for any fixed number whatever, it may have all possible values at pleasure. It cannot, however, have more than one value at a time. Its several values must be taken in succession.

In like manner a may represent any kind of quantity, as pounds, dollars, men, etc., but always in succession. The moment we attribute a specific numerical value to it, it ceases to be an algebraic quantity for that particular value, and becomes an arithmetical quantity. Whatever is here said of a, is equally true of any other quantities, b, c, d, etc.

3. The final letters of the alphabet (x, y, z, etc.) are used to denote quantities to be determined. They are called *unknown quantities ;* or, when they admit of an indefinite number of values, in succession, they are called *variables.*

Unknown quantities always depend for their values upon certain other quantities; for example, if we say x shall be a quantity whose third part shall always be equal to 5, the value of x will depend upon 3 and 5, and, although unknown for a moment, readily becomes known.

If we should take two quantities, x and y, and say that their product shall always be equal to a fixed number, as 100, x may be 5 and y 20 ; or x 4 and y 25, etc. Here x and y may have any number of *relative,* but never any *absolute,* values. In such a case they are called *variables.* The difference between an arbitrary quantity and a variable is, that all arbitrary quantities may be assumed at the same time and independently of each other; while variables may be assumed only in relation to each other. It is manifest that x and y, in the above example, could not both be assumed at the same time, and without regard to their relative values.

4. Since quantities, as long as they remain purely algebraic, have no fixed numerical value, it is impossible to perform any arithmetical operation upon them; such as addition, multiplication, etc. All that can be done, therefore, is to indicate such combinations as it may be desired to make. For this purpose certain signs are employed. The sign $+$ indicates *addition;* $-$, *subtraction;* $\times$, *multiplication;* $\div$, *division;* $=$, *equality.*

If a represents the number of days one man works, and b the number another works, since a may have any value, and b any value, we manifestly cannot tell how great their sum must be; but we can write them so as to show that their sum must be taken, if their values ever become fixed; thus, $a+b$ (read, a *plus* b) indicates that these quantities must be added together, when their numerical values are determined upon. In algebraic language they are said to be already added; and algebra knows nothing of any other kind of addition. All *actual* additions must be made by the laws of arithmetic. The same may be said of all other operations upon numbers.

To show that a quantity is to be subtracted from another, we write the quantity to be subtracted after that from which it is to be taken with the sign $-$ between; thus $a-b$ (read, a *minus* b) shows that b is to be taken from a, or, as is commonly said, b is subtracted from a.

To multiply quantities algebraically, we simply connect them together by the sign $\times$; thus, $a\times b\times c$ (read, a *multiplied by* b, *multiplied by* c) shows that their product is required.

It is far more usual, however, to write the quantities together, thus, abc, without any sign between. Sometimes dots are used; thus, $a\cdot b\cdot c$. When numbers are to be multiplied together the sign $\times$ must be used; thus, 4×5.

To show that one quantity is to be divided by another, we may write them thus, $a\div b$ (read, a *divided by* b). It is much more common, however, to write the divisor under the dividend, thus, $\frac{a}{b}$.

The sign $=$ is placed between quantities to show that they are equal; thus, $a=b$ (read, a *equal to* b) shows that an equality subsists between these quantities. The quantity to the left of the sign $=$ is called the *First Member;* that to the right, the *Second Member.*

5. When it is desired to show that two or more quantities, already connected by the signs plus or minus, are to be considered as a single quantity, the sign ( ), called a *parenthesis*, is used; thus, $(a+b-c)$ shows that the several quantities within are not to be separated, and are to be considered as a single quantity so long as the parenthesis remains.

This sign often takes the form [ ], or { }, called *brackets.* A bar over several quantities, thus, $\overline{a+b-c}$, means the same thing. Also the quantities may be written thus $\left.\begin{array}{r} +a \\ -b \\ +c \end{array}\right| x$; here the quantities on the left of the *bar* are to be severally multiplied by the quantity, x, on the right.

6. The sign $>$, indicates *inequality*.

The opening is turned towards the greater quantity; $a > b$ (read, a *greater than* b), shows that a is greater than b; $a < b$, shows that a is less than b.

7. The sign $\propto$, shows that quantities connected by it *vary together;* thus, $a \propto b$ means that a and b increase or decrease together.

8. The sign $\therefore$, is an abbreviation for *therefore, hence,* or *consequently;* $\because$ means *since* or *because.*

9. An *Algebraic Expression* is any quantity or combination of quantities written in algebraic language.

Thus, $\frac{a}{b}$ is an expression for the quotient of any two quantities; $(a+b)c$ is an expression for the product of the sum of two quantities by a third quantity.

10. *A Factor* is any quantity which enters an expression as a *multiplier.*

Thus, in the product abc, a, b, and c are each factors; so also ab, bc, and ac are factors of the same expression. Unity enters every expression as a factor. A factor may be made up of several quantities connected by the sign $+$ or $-$; thus, in $(a+b)c$ and $(a+b)(a-b)$, the quantities $(a+b)$ and $(a-b)$ are *factors.* In fractional expressions unity divided by the denominator, or by any factor of the denominator, is a factor of the expression; thus, in $\frac{a}{b}$, $\frac{1}{b}$, is a factor of $\frac{a}{b}$; in $\frac{c}{3ab}$, $\frac{1}{3}$, $\frac{1}{a}$, and $\frac{1}{b}$ are each factors.

11. *A Co-efficient* (co-factor) is any factor of an expression.

If a numerical factor enters an expression, that factor is usually written first, and is especially spoken of as the co-efficient. Thus, in the expressions $3ab$, $\frac{2}{3}cd$, and $\frac{a-c}{2}$; 3, $\frac{2}{3}$, and $\frac{1}{2}$ would be called the co-efficients, respectively.

Literal factors, however, are often called co-efficients; thus, in ax, $\frac{a}{b}\cdot x$, and bcy; a, $\frac{a}{b}$, and bc, would be called co-efficients.

So, also, numerical and literal factors taken together may be spoken of as co-efficients; thus, in the expressions $3ax$, $\frac{2ay}{c}$, and $3(a-b)z$; $3a$, $\frac{2a}{c}$, and $3(a-b)$ would be called the co-efficients of the unknown quantities, respectively.

Since multiplication is but abbreviated addition, each factor of an expression shows how many times all the other factors enter it additively; thus, in the expression $3ab$, the 3 shows that ab enters the quantity three times by addition, so that it may be written $ab+ab+ab$.

So if we had $a+a+a+a$ we could evidently write $4a$ instead. $\frac{a}{b}+\frac{a}{b}+\frac{a}{b}$ may be written $3\frac{a}{b}$ or $\frac{3a}{b}$. $\frac{a}{b}+\frac{a}{b}-\frac{a}{b}+\frac{a}{b}$ must be equal to $\frac{2a}{b}$.

If we had $(a+b)+(a+b)+\cdots\cdots$ written c times, we could write $c(a+b)$. Here c or $(a+b)$ may, either of them, be regarded as the co-efficient.

12. An expression composed altogether of simple factors is called a *monomial.*

By *simple factors* is meant those which are composed without the aid of the signs $+$ or $-$.

ab, $\frac{ab}{c}$, $3ab$, are monomials. A monomial is spoken of as a *term*, especially when found connected with other quantities by the sign plus or minus; thus, in the expression $7ax-\frac{ab}{cd}+g$, any one of the three monomials which enter it is a *term*. Sometimes complex expressions tied together by the parenthesis, or otherwise, are called *terms;* thus, in $a-(b+c)$ and $\frac{a+b}{c}-\frac{c}{b-c}$, the expressions $(a+c)$, $\frac{a+b}{c}$ and $\frac{c}{b-c}$ may be called terms.

13. *A Binomial* is an expression composed of *two terms;* a *Trinomial* has *three* terms. $a+b$, $6ab-3c$, $\frac{a}{b}-\frac{4b}{ac}$, are binomials. $ac+\frac{d}{b}-g$, and $4a-5c+\frac{3d}{b}$, are trinomials. A *Polynomial* is an expression composed of two or more terms; thus, $a+b$, $5ab+\frac{2}{a}-d$, $a+b+c-\frac{a}{b}+$ etc., are polynomials.

14. When the same factor enters an expression more than once, as in $3aaaabbccc$, the expression can be greatly shortened by writing any such factor but once, with a small figure to the right, and a little above, to show the number of times the quantity enters as a factor; thus the above expression would take the form, $3a^4b^2c^3$, read, *three* a *to the fourth power,* b *to the second power,* c *to the third power.* The small figure so used is called an *Exponent* or Index, because it shows the number of times the quantity, to which it is affixed, enters as a factor.

It is important to distinguish clearly between exponents and numbers which enter an expression as factors, called co-efficients. In the above example, the 3 written first is a multiplier, whereas the small figures do not enter the expression as quantities at all; they are merely signs, to show how many times other quantities, namely a, b, and c, enter as factors.

An exponent, therefore, is *any quantity used* to show how many times another quantity must be taken as a factor.

Letters are often used as exponents; thus, a^x, 5^a, $(a+b)^c$, $a^{\frac{1}{2}}$, $6^{\frac{m}{n}}$, (read, a *to the* x *power*, etc., a *to the one-half power*, b *to the* m *divided by* n *power*.) Any symbol of quantity may be used for the same purpose.

15. Terms are said to be *Homogeneous*, when they contain the same number of literal factors; thus, $9a^2b^3$, $2cx^4$, $25x^3y^2$, $abcxy$, are all homogeneous. They are said to be homogeneous with respect to a certain quantity or class of quantities, when there are the same number of such quantities in each; thus, ax^2, b^2cxy, $4ay^2$, are homogeneous with respect to the unknown quantities which enter them.

A polynomial or an equation is homogeneous when all of its terms are homogeneous.

16. *The Reciprocal* of any quantity is unity divided by that quantity.

Thus, $\frac{1}{a}$, $\frac{1}{a-b}$, $\frac{1}{a^2b}$, are the reciprocals of a, $a-b$, and a^2b.

17. *The Square Root* of a quantity is indicated by placing over it the sign $\sqrt{\ }$; when any other root is to be taken, a small number is written to the left and above the sign, to show the degree of the root required; thus, $\sqrt[3]{\ }$, indicates the third root; $\sqrt[4]{\ }$, the fourth root; $\sqrt[m]{\ }$, the mth root, and the sign is used thus, $\sqrt{a}$, $\sqrt[3]{a^2b^3}$, $\sqrt[5]{a^2b+d}$, $\sqrt{\frac{a+b}{a-b}}$. The small figure used to show the degree of the root required, is commonly called the *index*.

This sign grew out of the custom of the older algebraists of writing r, signifying *root*, before the quantity whose root was required. When used before an expression tied together by a bar, thus, $r \cdot \overline{a+b}$, it would take very nearly the form of the present radical sign.

SECTION II.

ALGEBRAIC TERMINOLOGY.

18. Algebra has a language of its own, which must be thoroughly mastered before the subject can be at all comprehended. In the foregoing Section we have, so to speak, learned the alphabet. We may now proceed to use it. Let it be borne well in mind that it is all pure conventionality. Any other signs or symbols might have been used; but now that we have agreed to use those already explained, we must adhere to them strictly.

19. First let us translate from English into Algebraic symbols.

1. *Add together any two quantities.*

Explanation.—Since they must be *any* two quantities, we cannot take particular numbers; and since numbers from their nature are always definite, we cannot take numbers at all. a and b are two such quantities, and since their values must remain undetermined, we can only write, $a+b$. Any other letters would have done as well.

2. Write the difference of any two quantities. *Ans.* $a-b$.
3. Write the product of any two quantities. *Ans.* ab.
4. Write the quotient of any two quantities. *Ans.* $\frac{a}{b}$.
5. Write the product of the sum and difference of two quantities. *Ans.* $(a+b)\ (a-b)$.
6. Write the quotient of the sum and difference. *Ans.* $\frac{a+b}{a-b}$.

Remark.—When a quantity has 2 for an exponent it is often read *square*, and though, perhaps not strictly correct, it is convenient to retain the custom; thus a^2 may be read *a square*. When a polynomial has 2 for an exponent, as $(a+b)^2$, it must be read a plus b *squar*ED. In like manner a^3 may be read *a cube*; $(a+b)^3$, $(a+b)$ *cub*ED.

7. Write the sum of the squares of two quantities. *Ans.* a^2+b^2.
8. The square of the sum. *Ans.* $(a+b)^2$.
9. The square of the difference.. *Ans.* $(a-b)^2$.
10. The square of the quotient. *Ans.* $\left(\frac{a}{b}\right)^2$.
11. The sum of the square roots. *Ans.* $\sqrt{a}+\sqrt{b}$.
12. The square root of the sum. *Ans.* $\sqrt{a+b}$.
13. The product of the square roots. *Ans.* $\sqrt{a}\times\sqrt{b}$, or $\sqrt{a}.\sqrt{b}$.

14. The square root of the product. *Ans.* $\sqrt{ab}$.

15. The quotient of the square roots. *Ans.* $\dfrac{\sqrt{a}}{\sqrt{b}}$.

16. The square root of the quotient. *Ans.* $\sqrt{\dfrac{a}{b}}$

17. The square root of the product of the squares.
Ans. $\sqrt{a^2 b^2}$.

18. The square root of the product of the sum and difference.
Ans. $\sqrt{(a+b)\,(a-b.)}$

19. The cube root of the product of the cubes. *Ans.* $\sqrt[3]{a^3 b^3}$.

20. Translate the following expressions into English:

1. $a-b,\ a+b,\ a^2+b^2,\ ab,\ a^2b^2,\ \dfrac{a}{b}$.

2. $a^2-b^2,\ (a-b)^2,\ a^2+b^2,\ (a+b)^2,\ (a+b)\,(a-b)$.

3. $\dfrac{a^2}{b^2},\ \dfrac{a+b}{a},\ \dfrac{a-b}{b},\ \dfrac{a+b}{a-b},\ \dfrac{(a+b)^2}{a^2+b},\ \dfrac{a-b}{(a-b)^2}$.

4. $\sqrt{a}+\sqrt{b},\ \sqrt{a},\ \sqrt{b},\ \sqrt{ab},\ \dfrac{\sqrt{a}}{\sqrt{b}},\ \sqrt{\dfrac{a}{b}},\ \sqrt{a-b},\ \sqrt{a^2+b^2}$.

5. $\sqrt{(a+b)(a-b)},\ \sqrt{\dfrac{a+b}{a-b}},\ \sqrt{\dfrac{a+b}{a}},\ \sqrt{\dfrac{a-b}{b}},\ \sqrt{\dfrac{a^2}{b^2}}$.
$\sqrt{\left(\dfrac{a}{b}\right)^2}$.

6. $\sqrt[3]{a^2},\ \sqrt[3]{a^2+b^2},\ \sqrt[3]{a^2b^2},\ \sqrt[3]{(a+b)^2},\ \sqrt[3]{a^2-b^2},\ \sqrt[3]{a^2-b^2},$
$\sqrt[3]{\dfrac{a}{b}},\ \sqrt[3]{\dfrac{a^3}{b^3}}$.

21. Translate the following into algebraic language:

1. The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first by the second, plus the square of the second. *Ans.* $(a+b)^2 = a^2+2ab+b^2$.

2. The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first by the second, plus the square of the second. *Ans.* $(a-b)^2=a^2-2ab+b^2$.

3. The product of the sum and difference of two quantities is equal to the difference of their squares. *Ans.* $(a+b)(a-b)=a^2-b^2$.

4. The quotient of the difference of the squares of two quantities by the difference of the quantities, is equal to the sum of the quantities.

Ans. $\dfrac{a^2-b^2}{a-b}=a+b.$

5. The quotient of the difference of the squares of two quantities by the sum of the quantities is equal to the difference of the quantities.

Ans. $\dfrac{a^2-b^2}{a+b}=a-b$

6. The product of the sum of two quantities by the first is equal to the square of the first, plus the product of the first by the second.

Ans. $(a+b)\,a=a^2+ab.$

Let the algebraic expressions in this article be translated back into English.

22. Translate the following expressions into English:

1. $\sqrt{a}\times\sqrt{b}=\sqrt{ab}.$

Ans. The product of the square roots of two quantities is equal to the square root of the product.

2. $\dfrac{\sqrt{a}}{\sqrt{b}}=\sqrt{\dfrac{a}{b}},\quad \left(\dfrac{\sqrt{a}}{\sqrt{b}}\right)^2=\dfrac{a}{b}.$

3. $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b.$

4. $(\sqrt{a}+\sqrt{b})^2=a+2\sqrt{ab}+b.$

5. $(\sqrt{a}-\sqrt{b})^2=a-2\sqrt{ab}+b.$

6. $\sqrt{a}=a^{\frac{1}{2}},\quad \sqrt[3]{a}=a^{\frac{1}{3}},\quad \sqrt[4]{b}=b^{\frac{1}{4}},\quad \sqrt[3]{a^2}=a^{\frac{2}{3}}.$

7. $\sqrt[n]{a}=a^{\frac{1}{n}},\quad \sqrt[n]{a^m}=a^{\frac{m}{n}},\quad \sqrt[n]{a^{-m}}=\dfrac{1}{a^{\frac{m}{n}}}.$

8. $\sqrt[m]{\sqrt[n]{a}}=\sqrt[mn]{a},\quad \sqrt[mn]{a}=a^{\frac{1}{mn}},\quad \sqrt[m]{\sqrt[n]{a^{-1}}}=\sqrt[mn]{\dfrac{1}{a}}.$

9. $(a+b)^{\frac{1}{2}}=\sqrt{a+b},\quad [(a+b)^2]^{\frac{1}{2}}=a+b.$

10. $(a^{\frac{1}{2}}+b^{\frac{1}{2}})(a^{\frac{1}{2}}-b^{\frac{1}{2}})=a-b.$

23. Express in algebraic symbols the following:

1. Divide a certain quantity into three equal parts.

Ans. $\dfrac{a}{3},\ \dfrac{a}{3},\ \dfrac{a}{3}.$

2. Divide a given quantity into two parts, one of which shall be three times the other.

Ans. $\frac{a}{4}, \frac{3a}{4}$.

3. What two numbers are those which differ from each other by a?

Ans. x and $x-a$; or, x and $x+a$.

4. The a part of a quantity added to its b part is equal to m.

Ans. $\frac{x}{a}+\frac{x}{b}=m$.

5. Three times a certain number minus that number is equal to $\frac{2}{3}$ the number less a. *Ans.* $3x-x=\frac{2}{3}x-a$.

6. The sum of two numbers is to the difference of those numbers as m is to n. *Ans.* $a+b : a-b :: m : n$.

7. The ratio of the square of the sum of two numbers to the sum of the squares of those numbers is as m is to n.

Ans. $\frac{a^2+b^2}{(a+b)^2}=\frac{n}{m}$.

24. If at any time particular values are given to the quantities in an expression, the indicated operations may then be performed. A result so found is called the *numerical* value of the expression. For each new set of values for the several quantities which enter an expression, there will be a new result as the numerical value of such expression.

The numerical value of the expression $ab-\frac{a}{b}$, when $a=4$, and $b=2$, gives $4\times 2-\frac{4}{2}=6$, which is the value of the expression in this case.

If $a=7$, and $b=5$, we should have $7\times 5-\frac{7}{5}=\frac{168}{5}$.

Find the numerical values of the following, letting $a=1$, $b=2$, $c=3$, and $d=4$:

1. $3ab-\frac{cd}{b}$, $\frac{ab}{2}+\frac{a-b}{4cd}$, $\frac{(a+b)^2}{cd}\times\frac{2d}{a}$.

2. $(a+b)(a+b)$, $\frac{a+b}{5ab}\div\frac{d-a}{2c}$, $[(a+b)(c-a)]d$.

3. $\sqrt{d}\,[a(b+c)]\sqrt{a}$, $\frac{\sqrt{b+a}}{\sqrt{d}}\times\frac{\sqrt{d-c}}{\sqrt{a}}$, $\frac{abcd}{\sqrt{b-a}}\times\frac{\sqrt{c-b}}{abcd}$.

4. $\frac{5b-3}{4c-a}\times\frac{1}{b}$, $\frac{1}{c}\left(\frac{a}{b}+\frac{c}{d}\right)\left(\frac{a}{c}+\frac{b}{a}\right)$, $\frac{(a+b-2)(c-a+d)}{\left(\frac{a}{b}+\frac{c}{d}\right)\left(\frac{a}{b}-\frac{a}{c}\right)}$.

SECTION III.

TREATMENT OF THE + AND − SIGNS.

25. All elementary operations of Algebra fall under one of the two following heads, namely:

I. *Indications.*

II. *Transformations.*

As we have seen in the previous section, *Indications* are the operations of expressing in algebraic language whatever may be stated in spoken language.

Transformations comprise all changes which may be lawfully performed upon expressions after they are written in algebraic symbols.

The simplest transformations are those made upon quantities connected by the signs *plus* or *minus*.

Thus, $a+a+a+a$, may evidently be written $4a$. This is a change of form without a change in the numerical value.

$a+a-a+a=2a$, is another such change.

Algebraists have commonly called these transformations *addition;* but manifestly the algebraic sum is altogether accomplished in the operation of Indication. The quantities are no more added after the form is changed than when they are simply connected together by their proper signs.

26. The Nature of the Signs + and −.

Let us now try to understand the nature of the signs + and −.

Their first and most obvious use is to connect quantities together so as to show that they are to be added or subtracted. But, then, certain quantities may, from their nature, be additive, while others are subtractive, before any combination takes place. For example, a man, casting up his accounts, would consider all amounts due him to be augmentative, while all amounts which he owed would be diminutive of his capital. The first class would take the sign +; the second the sign −. It is, however, a mere matter of agreement as to which shall be called *plus* and which *minus;* but they are always contrary the one to the other, so that the establishment of either determines the other. Quantities affected with the sign + are called *Positive;* those with the sign − are called *Negative.*

To illustrate this, suppose we take a right line, ab, and agree to reckon all distances

a o b

to the right of the point o as positive; then, of necessity, those to the left would be negative. If we had determined to reckon distances to the left of o positive, those to the right would have been negative, and so in general.

The sign of a quantity must be known before we can use it in combination with other quantities. *Plus* is always understood where no sign is written. A quantity with the signs + and —, thus $\pm a$, is to be used first positively and then negatively.

27. To find the algebraic sum of several quantities.

When quantities are given with their respective signs, to find their aggregate, or, as it is called, their *algebraic sum*, we have this simple principle:

Write the several quantities one after the other in any order, connected by their respective signs.

EXAMPLES.

1. Add, $3ab$, $4a^2b$, $2ab$, $-ab$, and $-5a^2b$.

Ans. $3ab+4a^2b+2ab-ab-5a^2b$.

2. Add, $-\frac{1}{2}a$, $\dfrac{b-c}{d}$, $-bc$, $a^{\frac{1}{2}}b^m$, and $\sqrt{a}$.

Ans. $\dfrac{b-c}{d}-\frac{1}{2}a-bc+a^{\frac{1}{2}}b^m+\sqrt{a}$.

Remark.—It is usual to place a positive quantity at the beginning.

3. Add, $\dfrac{(a+b)c}{2}$, 24, $-\sqrt{67a}$, $\dfrac{1}{14b}$, and $(a+b)(a-b)$.

Ans. $\dfrac{(a+b)c}{2}+24-\sqrt{67a}+\dfrac{1}{14b}+(a+b)(a-b)$.

4. Add, $-\dfrac{a}{b}$, $\dfrac{2a}{b}$, $-\dfrac{5a}{b}$, 1, $\frac{1}{2}$, $\left(\dfrac{a}{b}\right)^m$, and $\left(\dfrac{a}{b}\right)^{m+n}$.

Ans. $\dfrac{2a}{b}-\dfrac{a}{b}-\dfrac{5a}{b}+1+\frac{1}{2}+\left(\dfrac{a}{b}\right)^m+\left(\dfrac{a}{b}\right)^{m+n}$

5. Add, $7a^2-2ab^{\frac{1}{2}}$, $3a^2+4ab^{\frac{1}{2}}$, and $2a-a^m+5a^{\frac{1}{3}}$.

Ans. $7a^2-2ab^{\frac{1}{2}}+3a^2+4ab^{\frac{1}{2}}+2a-a^m+5a^{\frac{1}{3}}$.

28. To reduce a polynomial to the least number of terms.

Terms are said to be *like* or *similar* when they contain the same letters, and the several letters have the same exponents, respectively. $2a^{\frac{1}{2}}b^mc^3$ and $5a^{\frac{1}{2}}b^mc^3$ are like terms. The algebraic sum of several quantities can often be much shortened by gathering into one all the terms which are alike.

For example, $3ab+4a^2b+2ab-ab-5a^2b$ may be written $4ab-a^2b$; for, $3ab$ and $2ab$, both being $+$, give $5ab$; but $-ab$ (that is, one ab to be taken away) leaves $4ab$; so $4a^2b$ less $5a^2b$, leaves one a^2b minus.

So in general, to reduce a polynomial to the smallest number of terms we may say:

Gather like terms together, giving the results their proper signs, respectively.

Great care must be taken to combine only terms whose literal parts are *entirely* the same, and to add and subtract the numerical parts according to their signs.

Examples.

Reduce the following polynomials to the smallest number of terms:

1. $2a-4a^2b^2+3a+5a^2b^2-4a$. *Ans.* $a+a^2b^2$.

2. $3b^{\frac{1}{2}}c-2b^{\frac{1}{2}}c+a^m-b^{m+1}+5a^m-2b^{m+1}$. *Ans.* $b^{\frac{1}{2}}c+6a^m-3b^{m+1}$.

3. $\frac{a}{b}+\frac{cd}{a}-g+2\frac{a}{b}-3\frac{cd}{a}+g+24-12$.

Ans. $\frac{3a}{b}-2\frac{cd}{a}+12$.

4. $3a^2c^{\frac{1}{2}}+5a^mb^n-a^2c^{\frac{1}{2}}+6a^{\frac{1}{3}}b+2a^2c^{\frac{1}{2}}-2a^mb^n-a^{\frac{1}{3}}b$.

Ans. $4a^2c^{\frac{1}{2}}+3a^mb^n+5a^{\frac{1}{3}}b$.

5. $a^2+2ab+b^2+a^2-2ab+b^2=2a^2+2b^2$.

6. $a+2\sqrt{ab}+b+a-2\sqrt{ab}+b=2a+2b$.

7. $\sqrt{a}+2\sqrt[4]{ab}+\sqrt{b}+\sqrt{a}-2\sqrt[4]{ab}+\sqrt{b}=2\sqrt{a}+2\sqrt{b}$.

8. $2(a+b)^{\frac{1}{2}}-3(a-b)^{\frac{1}{3}}+5(a+b)^{\frac{1}{2}}-2(a-b)^{\frac{1}{3}}$.

Ans. $7(a+b)^{\frac{1}{2}}-5(a-b)^{\frac{1}{3}}$.

9. $3\frac{\sqrt{a+b}}{\sqrt{c}}-\left(\frac{a-c}{bd}\right)^{\frac{1}{m}}-\frac{\sqrt{a+b}}{\sqrt{c}}+\left(\frac{a-c}{bd}\right)^{\frac{1}{m}}+\frac{a^{m-1}}{b^{n-1}}$.

Ans. $\frac{2\sqrt{a+b}}{\sqrt{c}}+\frac{a^{m-1}}{b^{n-1}}$.

10. $25a^2b^3c+12a^mb^2-5a^2b^3c+a^mb^2-3a^2b^3c+5a^mb^2$.

Ans. $17a^2b^3c+18a^mb^2$.

29. Subtraction of quantities.

To subtract a positive quantity, as $+b$ from a, we should, strictly, write $a-(+b)$, using the parenthesis to prevent the confusion of signs; but since $+b$ is but another way of writing b itself, we may drop the + sign, and the parenthesis with it, and write $a-b$. When the quantity to be subtracted is already negative, as $-b$, the indication is made in the same way, $a-(-b)$; but we cannot, as before, drop the — sign of the quantity, since, in that case, we should have for a result, $a-b$ as before, and should thus have taken away, not $-b$, but $+b$. To take away a negative *quantity* is really to augment the quantity from which it is said to be taken; thus, $(25-5)$ is 20; now from $(25-5)$ remove the -5, that is, strike it out, and, of course, the 25 will be left. By taking away -5 we have really added 5 to the quantity $(25-5)$, from which it was taken; thus, $25-5-(-5)=25-5+5=25$, or in general, $a-b-(-b)=a-b+b=a$.

A man who is insolvent owes, say, a dollars. It would be the same thing to say, he has $-a$ dollars due him. To subtract or take away any part of his debts, is really to augment his fortune by just that amount; and so in general,

To subtract a negative quantity, is to add the numerical value of the quantity; or practically,

To subtract any quantity, write it after the quantity from which it is to be taken with its sign changed.

EXAMPLES.

1. From a take b, ab, $-d$, $3b$, $-2ab$, $+d$, and then simplify the resulting expression.

Ans. $a-b-ab+d-3b+2ab-d=a-4b+ab$.

2. From $a+b$ take b, $-c^2$, $3c$, $-2b$, and simplify.

Ans. $a+b-b+c^2-3c+2b=a+2b+c^2-3c$.

3. From $3.\frac{a}{b}-cd$ take $\frac{a}{b}-cd$ and $\frac{a^m}{b^m}$.

Ans. $3\frac{a}{b}-cd-\frac{a}{b}+cd-\frac{a^m}{b^m}=2\frac{a}{b}-\frac{a^m}{b^m}$.

4. From $a+\sqrt{ab}+b$ take $a-\sqrt{ab}$ and $-b$.

Ans. $a+\sqrt{ab}+b-a+\sqrt{ab}+b=2\sqrt{ab}+2b$.

5. From $a^{\frac{1}{2}}-b^{\frac{1}{2}}$ take $2a^{\frac{1}{2}}$, $-3b^{\frac{1}{2}}$, $-5a^{\frac{1}{2}}$, and $b^{\frac{1}{2}}$.

Ans. $a^{\frac{1}{2}}-b^{\frac{1}{2}}-2a^{\frac{1}{2}}+3b^{\frac{1}{2}}+5a^{\frac{1}{2}}-b^{\frac{1}{2}}=4a^{\frac{1}{2}}+b^{\frac{1}{2}}$.

6. From $2\sqrt{a}+3\sqrt{b}$ take $-\sqrt{a}$, $2\sqrt{b}$, $\frac{1}{2}\sqrt{a}$, and $-\sqrt{b}$.

Ans. $2\sqrt{a}+3\sqrt{b}+\sqrt{a}-2\sqrt{b}-\frac{1}{2}\sqrt{a}+\sqrt{b}=\frac{5}{2}\sqrt{a}+2\sqrt{b}$.

7. From $\dfrac{a-b}{c}$ take $-3\cdot\dfrac{a-b}{c}$, $4.\dfrac{(a+b)}{c}$, and $-\frac{1}{3}\cdot\dfrac{a-b}{c}$.

Ans. $\dfrac{a-b}{c}+3\cdot\dfrac{a-b}{c}-4\dfrac{(a-b)}{c}+\frac{1}{3}\cdot\dfrac{a-b}{c}=\frac{1}{3}\cdot\dfrac{a-b}{c}$.

8. From $\sqrt[3]{ab}-c$ take $-2\sqrt[3]{ab}$, $5c$, $\dfrac{\sqrt[3]{ab}}{2}$ and $-6c$.

Ans. $\sqrt[3]{ab}-c+2\sqrt[3]{ab}-5c-\dfrac{\sqrt[3]{ab}}{2}+6c=\frac{5}{2}\sqrt[3]{ab}$.

9. From $\sqrt[m]{a}-2a\sqrt[n]{b}$ take $9\sqrt[m]{a}$, $-2a\sqrt[n]{b}$, and $-8\sqrt[m]{a}$.

Ans. $\sqrt[m]{a}-2a\sqrt[n]{b}-9\sqrt[m]{a}+2\sqrt[n]{b}+8\sqrt[m]{a}=0$.

30. Subtraction of polynomials.

To indicate the subtraction of any polynomial, we have but to write such polynomial within a parenthesis, and connect it by the $-$ sign with the quantity from which it is to be taken; thus, to take $a+b-c$ from d, we have $d-(a+b-c)$.

Now the minus sign here means that we are first to get the algebraic sum of all the quantities within the parenthesis, and then take this sum from d. Manifestly the same thing can be accomplished by taking, first a, and then b, and then $-c$, away from d; thus, $d-a-b+c$. The parenthesis has disappeared, and the several quantities have changed their signs. We may then say, that,

To subtract a polynomial from any quantity, we have but to write the several terms in succession after the quantity from which it is to be taken with their signs changed.

EXAMPLES.

1. From $3a-2b$ take $a+4b$ and simplify.

Ans. $3a-2b-a-4b=2a-6b$.

2. From a^n-a take $a^m+a^n-a^2$.

Ans. $a^n-a-a^m-a^n+a^2=a^2-a^m-a$.

3. From $a^{\frac{m}{n}}b^{-2}+1$ take $-3a^{\frac{m}{n}}b^{-2}+1$.

Ans. $a^{\frac{m}{n}}b^{-2}+1+3a^{\frac{m}{n}}b^{-2}-1=4a^{\frac{m}{n}}b^{-2}$.

4. From $8a^{p-1}c^q$ take $3a^{p-1}c^q+a^{-\frac{m}{n}}c^{q+p-1}-\sqrt{a^{\frac{m}{n}}}$.

Ans. $8a^{p-1}c^q-3a^{p-1}c^q-a^{-\frac{m}{n}}c^{q+p-1}+\sqrt{a^{\frac{m}{n}}}=5a^{p-1}c^q-a^{-\frac{m}{n}}c^{q+h-1}+\sqrt{a^{\frac{m}{n}}}$.

5. Simplify $a^{\frac{1}{2}}-b^m-(2a^{\frac{1}{2}}-b^m-1)$.

Ans. $a^{\frac{1}{2}}-b^m-2a^{\frac{1}{2}}+b^m+1=1-a^{\frac{1}{2}}$.

6. Simplify $a^2-7a^2b^3c-\frac{1}{2}-(a^2-7a^2b^3c+\frac{1}{2})$.

Ans. $a^2-7a^2b^3c-\frac{1}{2}-a^2+7a^2b^3c-\frac{1}{2}=-1$.

7. Simplify $x^2-x-1-(-x^2+x+1)$.

Ans. $x^2-x-1+x^2-x-1=2x^2-2x-2$.

Remark.—It will be observed that in removing a parenthesis from a polynomial when it has the minus sign before it, we change the signs of all the terms within. When the sign before the parenthesis is plus, the parenthesis may be removed without any other change.

Remove the parentheses and simplify the following:

8. $a-(-b+c-a)=a+b-c+a=2a+b-c$.

9. $1+(a-b-1)=1+a-b-1=a-b$.

10. $a-(-a)=2a$; $a-(-1)=a+1$.

11. $3a^2b-\frac{a}{b}-\left(-3a^2b+\frac{a}{b}\right)=6a^2b-\frac{2a}{b}$.

12. $a^{\frac{1}{2}}-1+(-a^{\frac{1}{2}}+1)=0$.

13. $-\frac{1}{2}+1-(-\frac{1}{2}+1+\sqrt{c})=-\sqrt{c}$.

14. $a^{m-1}-b^{\frac{m}{n}}-(a^{m-1}-b^{\frac{m}{n}}+1)=-1$.

15. $1-\frac{1}{2}a+(1-\frac{1}{2}a)=2-a$.

16. $a-(-a)-(-a)-(2a)=a$.

Remark.—We may put a parenthesis upon the algebraic sum of several quantities without any other change, provided that the *positive sign* stands before the parenthesis; but if the negative sign occupies that place, *the signs of all the terms within must be changed.*

Enclose the following in parentheses, first with a + sign, and then with a − sign.

1. $a-b+c=(a-b+c)$
$\quad\quad\quad\;=-(-a+b-c)$.

2. $\frac{1}{2}a-2c+\frac{a}{b}=\left(\frac{1}{2}a-2c+\frac{a}{b}\right)$

$$=-\left(-\tfrac{1}{2}a+2c-\frac{a}{b}\right).$$

3. $a=(a)$; $1-a=(1-a)$; $a^m=(a^m)$
$=-(-a)$, $1-a=-(-1+a)$, $=-(-a^m)$.

Remark.—Where there are several parentheses or brackets, one within another, in removing them begin by removing the outer one, and then the next, and so on. We may, however, begin with the inner one.

Examples.

Remove the brackets from the following:

1. $-[a-(a+b-1)]=-a+(a+b-1)=b-1.$

2. $a-[-\{-(a+b)\}]=a+\{-(a+b)\}=a-(a+b)=a-a-b=-b$

3. $1-[a-(a+b)+(a-b)+b]=1-a+(a+b)-(a-b)-b=1-a$
$+a+b-a+b-b=1-a+b.$

4. $1-\{+[-(-1)]\}=0.$

5. $-(1+(-1))=0.$

The management of fractional quantities in Algebra and Arithmetic is entirely the same.

Now, since we can separate a fraction, such as $\frac{10}{11}$ into as many partial fractions as we please to break the numerator into parts, thus $\frac{2}{11}+\frac{3}{11}+\frac{5}{11}$, we may do the same thing with an algebraic fraction. For example, $\frac{a+b-c}{d}$, may be written, $\frac{a}{d}+\frac{b}{d}-\frac{c}{d}$.

Now, if such a fraction as this, having the algebraic sum of two or more quantities for its numerator, has the $-$ sign before it, and it should be thus broken into parts, the removal of the division bar will act altogether like the removal of a parenthesis. The place of the bar must, therefore, be supplied by a parenthesis; or the signs of the partial fractions must all be changed.

Examples.

Separate the following into partial fractions:

1. $-\frac{a+b-c}{d}=-\left(\frac{a}{d}+\frac{b}{d}-\frac{c}{d}\right)=-\frac{a}{d}-\frac{b}{d}+\frac{c}{d}.$

2. $\dfrac{a+b-c}{d}=\dfrac{a}{d}+\dfrac{b}{d}-\dfrac{c}{d}$.

3. $-\dfrac{2a^2b-c^m}{3a^nc^2}=-\left(\dfrac{2a^2b}{3a^nc^2}-\dfrac{c^m}{3a^nc^2}\right)=-\dfrac{2a^2b}{3a^nc^2}+\dfrac{c^m}{3a^nc^2}$.

4. $\dfrac{-a^{\frac{1}{2}}+b}{a+b}=-\dfrac{a^{\frac{1}{2}}}{a+b}+\dfrac{b}{a+b}$.

Remark.—The sign — standing before the division bar shows that the whole fraction is to be subtracted; or, in other words, that, after all the operations indicated are performed, the result is to be subtracted; but when the sign stands before the first term of the numerator, as in the last example, it affects only that term.

5. $\dfrac{-5+4a-2^{\frac{1}{2}}}{5^m}=\dfrac{-5}{5^m}+\dfrac{4a}{5^m}-\dfrac{2^{\frac{1}{2}}}{5^m}$.

6. $-\dfrac{-5+4a-2^{\frac{1}{2}}}{5^m}=-\left(-\dfrac{5}{5^m}+\dfrac{4a}{5^m}-\dfrac{2^{\frac{1}{2}}}{5^m}\right)=\dfrac{5}{5^m}-\dfrac{4a}{5^m}+\dfrac{2^{\frac{1}{2}}}{5^m}$.

7. $-\left(\dfrac{a+b}{a-b}-\dfrac{c}{d}\right)=-\left(\dfrac{a}{a-b}+\dfrac{b}{a-b}-\dfrac{c}{d}\right)=-\dfrac{a}{a-b}-\dfrac{b}{a-b}+\dfrac{c}{d}$.

8. $-\left[\dfrac{a+b}{(a+b)^2}-\dfrac{a-b}{(a+b)^m}\right]=$

$$-\left[\frac{a}{(a+b)^2}+\frac{b}{(a+b)^2}-\left(\frac{a}{(a+b)^m}-\frac{b}{(a+b)^m}\right)\right]=$$

$$-\frac{a}{(a+b)^2}-\frac{b}{(a+b)^2}+\frac{a}{(a+b)^m}-\frac{b}{(a+b)^m}.$$

31. Meaning of the terms SUM and DIFFERENCE.

The terms *Sum* and *Difference* must be understood in Algebra in their largest sense. To add does not necessarily mean to augment, nor does subtraction always mean a diminution; thus, $-b$ added to $+b$ gives 0; while $-b$ subtracted from $+b$ gives $2b$.

By the *sum* we are to understand the result obtained from connecting the quantities by their own signs; a *difference* results when certain terms are connected with their signs changed.

SECTION IV.

MONOMIALS—EXPONENTS AND THE SIGNS × AND ÷.

32. An Exponent is, by definition, a sign which shows how many times the quantity to which it is affixed is to be reckoned as a factor.

Exponents may be entire or fractional, positive or negative.

33. Entire and Positive Exponents.

5^3 is the same as $5\times5\times5$, or 125 : a^3 is but another way of writing $a\cdot a\cdot a$; $a^2b^3=a\cdot a\cdot b\cdot b\cdot b$; $(a+b)^2=(a+b)(a+b)$; $(\sqrt{a})^3=\sqrt{a}\cdot\sqrt{a}\cdot\sqrt{a}$; $a^m=a\cdot a\cdot a\cdot a\cdots\cdots$ written m times.

Quantities united by the sign × or ÷ are simply made co-factors; thus, $3a^2b\times5ab^3$ shows that the continued product of all the factors is required. The sign × may, however, be dropped altogether and the quantities be written together, where no confusion is like to result; thus, $3a^2b5ab^3$; but in such a case as this it is usual to retain the × or use the ·

Now the order in which factors are taken makes no difference in the product; so that we may write the above expression thus, $3\times5a^2abb^3$. It will be observed that the sign × must now be used between the numerals to prevent mistake.

But here we have the indicated multiplication of two factors which can be actually performed; so that 15 can be written in the place of 3×5.

We see, further, that a enters three times as a factor, so that a^2a may be written a^3. In like manner bb^3 is equal to b^4. It is thus manifest that $3a^2b\times5ab^3=15a^3b^4$.

We have simply multiplied the numerical factors together and written each letter once with an exponent equal to the sum of the several exponents of that letter. Transformations of this kind are called multiplication.

We may say, then, in general, that:

To multiply monomials together, multiply the numerical factors together, and after this product write the several letters which enter the monomials, giving to each an exponent equal to the sum of all the exponents of that letter in the several terms.

Examples.

1. Multiply $4a^2bc$ by $5abc^2$. *Ans.* $20a^3b^2c^3$.
2. Multiply a^5d^2f by $2ad^3f^2$. *Ans.* $2a^6d^5f^3$.
3. Multiply x^my by x^my. *Ans.* $x^{2m}y^2$.
4. Multiply $x^{m-1}y$ by xy^{m+1}. *Ans.* x^my^{m+2}.
5. Multiply $5ax^m$ by $3a^nx^n$. *Ans.* $15a^{n+1}x^{m+n}$.
6. Multiply $25a^pb^2c$ by $3a^3b^ac$. *Ans.* $75a^{p+3}b^{a+2}c^2$.
7. Multiply $\frac{1}{2}a^2b^3$ by $\frac{1}{2}ab^2$. *Ans.* $\frac{1}{4}a^3b^5$.
8. Multiply $\frac{2}{3}a^xy^n$ by $\frac{3}{4}a^yyz^2$. *Ans.* $\frac{1}{2}a^{x+y}y^{n+1}z^2$.
9. Multiply $a^mb^nc^p$ by abc. *Ans.* $a^{m+1}b^{n+1}c^{p+1}$.
10. Multiply $\frac{5}{7}a^{m+n}b$ by $\frac{2}{9}a^{m+n}b$. *Ans.* $\frac{10}{63}a^{2m+2n}b^2$.
11. Multiply $x^{m+n}b^{n+1}$ by $\frac{1}{2}x^{m-n}b^{1-n}$. *Ans.* $\frac{1}{2}x^{2m}b^2$.
12. Multiply $x^{m-n+1}b^{p-q}$ by $x^{n-m-1}b^{q-p}$. *Ans.* $x^\circ b^\circ$.

Remark.—We shall find that the same laws govern the management of fractional and negative exponents as when they are positive and entire, so that we may employ them in our examples at once. In finding the sum of fractional exponents, let the student remember to apply the rules for the addition or subtraction of fractions in arithmetic.

13. $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a$.

14. $abc \times a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}} = a^{\frac{3}{2}}b^{\frac{3}{2}}c^{\frac{3}{2}}$.

15. $\frac{1}{2}x^{\frac{1}{2}}y^{\frac{2}{3}} \times \frac{1}{3}x^my^{\frac{1}{3}} = \frac{1}{6}x^{m+\frac{1}{2}}y$.

16. $5x^3y^2z \times axy^{\frac{1}{2}}z^{\frac{1}{2}} = 5ax^4y^{\frac{5}{2}}z^{\frac{3}{2}}$.

17. $a^{-2}bc^{\frac{1}{2}} \times a^3b^{-2}c^{\frac{1}{2}} = ab^{-1}c$.

18. $25a^mb^{\frac{m}{n}} \times 5a^nb^{\frac{p}{q}} = 125a^{m+n}b^{\frac{m}{n}+\frac{p}{q}}$.

19. $a^mb^nc^p \times a^{-m}b^{-n}c = a^\circ b^\circ c^{p+1}$.

20. $\dfrac{a^{\frac{m}{n}}b^cx^{-1}}{5} \times \dfrac{a^pb^ac}{2} = \dfrac{a^{\frac{m}{n}+p}\,b^{a+c}cx^{-1}}{10}$.

34. The law of signs in Multiplication and Division.

So far no mention has been made of the signs of the terms multiplied together. Let us now inquire what effect these signs will have upon the product.

In the multiplication of two quantities we simply repeat one of them additively as many times as there are units in the other. Let us, then, multiply two quantities, a and b, together, and first let them both be positive. $+a$ must be added to itself until there are altogether as many times a as there are units in b, so that we shall have $+ab$.

Thus we see that,

The product of two positive quantities is positive.

Now, let a be negative and b be positive. $-a$ must now be added to $-a$ until $-a$ has been repeated b times. But $-a$ and $-a$ gives $-2a$; $-a$ and $-a$ and $-a$ gives $-3a$; and when taken b times we must have $-ba$.

If a had been positive and b negative, $-b$ would have been taken additively a times, and the result would have been again $-ab$. Thus we see that,

The product of a positive and negative quantity is always negative.

Now let us take a and b both negative. We shall have $(-a)(-b)$. But $(-a) = -(+a)$; and so we may write $-(+a)(-b) = -(-ab) = +ab$. Thus we see that,

The product of a negative quantity by a negative quantity is positive.

Since the product of the divisor and quotient must always produce the dividend, the same principles hold good in division.

The law of signs may then be summed up as follows:

In multiplication and division like signs give plus, and unlike minus.

Examples.

1. Multiply $5a^2b$ by $-7a^{\frac{1}{2}}b^2$. *Ans.* $-35a^{\frac{5}{2}}b^3$.
2. Multiply $-\frac{1}{2}a^mb^n$ by a^mb. *Ans.* $-\frac{1}{2}a^{2m}b^{n+1}$.
3. Multiply $-\frac{1}{2}a^mb^n$ by $-a^mb$. *Ans.* $\frac{1}{2}a^{2m}b^{n+1}$.
4. Multiply $\frac{1}{2}a^mb^n$ by $-5a^mb$. *Ans.* $-\frac{5}{2}a^{2m}b^{n+1}$.
5. Multiply $\frac{1}{2}a^mb^n$ by $\frac{2}{3}a^mb$. *Ans.* $\frac{1}{3}a^{2m}b^{n+1}$.
6. Multiply $-9a^{m+1}b^{-2}$ by $5ab^mc$. *Ans.* $-45a^{m+2}b^{m-2}c$.

7. Multiply $a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{m}{n}}$ by $-3a^{-\frac{1}{2}}b^{-\frac{1}{3}}c^{-\frac{m}{n}}$. *Ans.* $-3a^{\circ}b^{\circ}c^{\circ}$.

8. Multiply $-a^{\circ}b^{-1}c$ by -1. *Ans.* $a^{\circ}b^{-1}c$.

9. Multiply $ab^{\circ}c^{-1}$ by $a^{-n}bc$. *Ans.* $a^{1-n}bc^{\circ}$.

10. Multiply $ab^{\circ}c^{-1}$ by $ab^{-\circ}c$. *Ans.* a^2.

Remark.—When a quantity has an exponent zero, such exponent shows that the quantity does not enter as a factor at all. It may therefore be omitted altogether. Remember, however, that the quantity itself is not zero. Any quantity to the zero power is thus 1; but we shall see further upon this point.

11. $ab \times ab \times -ab = -a^3b^3$.

12. $-ax^{\frac{1}{2}} \times -ax^{\frac{1}{3}} \times ax^2 = a^3x^{\frac{17}{6}}$.

13. $-2a^mb^{\frac{2}{3}} \times -3ab \times -4a^{-m}b^{-\frac{2}{3}} = -24ab$.

14. $x^{\frac{m}{n}}y^{\frac{p}{q}} \times x^{\frac{p}{q}}y^{\frac{m}{n}} \times -x^{\frac{1}{2}}y^{\frac{1}{3}} = -x^{\frac{m}{n}+\frac{p}{q}+\frac{1}{2}}y^{\frac{p}{q}+\frac{m}{n}+\frac{1}{3}}$.

15. $10a^{m-n}b^{-p} \times -a^{\circ} = -10a^{m-n}b^{-p}$.

16. $a(-a)(-a)a(-a) = -a^5$.

17. $(-1)(-1)(-1)(-a^{\circ}) = 1$.

18. $(-a^{\circ})(-b^{\circ})(-c^{\circ}) = -1$.

19. $(\frac{1}{2})^{\circ}(\frac{1}{3})^{\circ}(\frac{m}{n})^{\circ}\sqrt{a} = \sqrt{a}$.

35. Operation upon Fractions.

It has already been said that an algebraic fraction does not differ from a numerical fraction in principle.

In the treatment, therefore, of algebraic fractions, we have but to apply the laws of arithmetic to those already established for the management of algebraic symbols.

Examples.

1. Multiply $\frac{a}{b}$ by $\frac{c}{d}$. *Ans.* $\frac{ac}{db}$.

2. Multiply $-\frac{a}{b}$ by $\frac{c}{d}$. *Ans.* $-\frac{ac}{bd}$.

3. Multiply $-\frac{a}{b}$ by $-\frac{c}{d}$. *Ans.* $\frac{ac}{bd}$.

4. Multiply $-\frac{a}{b}$ by $\frac{a}{b}$. *Ans.* $-\frac{a^2}{b^2}$.

5. Multiply $\frac{-a^{\frac{1}{2}}}{-b}$ by $\frac{ab}{c}$. *Ans.* $\frac{-a^{\frac{3}{2}}b}{-bc}=\frac{a^{\frac{3}{2}}}{c}$.

6. Multiply $-\frac{a^m}{b^3}$ by $\frac{b^2}{-a^2}$. *Ans.* $-\frac{a^m b^2}{-a^2b^3}=\frac{a^m}{a^2b}$.

7. Multiply $\frac{3x^{m+n}}{2y^{-1}}$ by $\frac{5x^{-n}}{y^5}$. *Ans.* $\frac{15x^m}{2y^4}$.

8. Multiply $\frac{5x^{\frac{m}{n}-1}y^p}{3ab^{-1}}$ by -1. *Ans.* $-\frac{5x^{\frac{m}{n}-1}y^p}{3ab^{-1}}$.

9. $\frac{a}{b}\times\frac{-a}{b}\times\frac{a}{-b}\times\frac{-a}{-b}=\frac{a^4}{b^4}$.

10. $\frac{abc}{-1}\times\frac{-1}{abc}=1$.

11. $\frac{a^n}{b^m}\times\frac{b^{-m}}{a^{-n}}\times-2=-2\frac{a^nb^{-m}}{a^{-n}b^m}$.

12. $\frac{a^\circ}{a}\times\frac{b}{b^\circ}\times\frac{a}{b}=1$.

36. The Powers of Quantities.

The *Power* of a quantity is the product obtained by using the quantity a certain number of times as a factor; thus, the second power of 3 is 9, the third power of 2 is 8. a^2 is the second power of a; a^3, the third power; a^m, the mth power of a.

To form the power of any quantity multiply the quantity by itself as many times as there are units in the exponent of the power less one.

A practical rule for finding the power of a monomial is:

Raise the numerical factor, if there be one, to the required power and multiply the exponent of each letter by the exponent of the power.

This process is called *Involution.*

EXAMPLES.

1. $(a^2)^2$, $(2a)^5$, $\left(-3x^{\frac{1}{2}}\right)^2$, $\left(-\frac{1}{2}x^{\frac{1}{2}}\right)^3$, $(-a^{-1})^5$, $(a^m)^n$, $\left(a^{\frac{m}{n}}\right)^p$.

Ans. a^4, $32a^5$, $9x$, $-\frac{1}{8}x^{\frac{3}{2}}$, $-a^{-5}$, a^{mn}, $a^{\frac{pm}{n}}$.

Remark.—It will be observed that the sign of an *even* power must *always be positive ;* and

That the sign of an *odd* power must always be the *same as the sign of the term itself.*

2. $\left(\frac{a}{b}\right)^2,\ \left(\frac{a}{b}\right)^5,\ \left(-a^2\right)^3,\ \left(-a^2\right)^4,\ \left(\frac{-a}{b}\right)^3,\ \left(-\frac{a}{b}\right)^4,\ \left(-\frac{a}{b}\right)^5.$

Ans. $\frac{a^2}{b^2},\ \frac{a^5}{b^5},\ -a^6,\ a^8,\ -\frac{a^3}{b^3},\ \frac{a^4}{b^4},\ -\frac{a^5}{b^5}.$

3. $\left(\frac{12a^2b^3}{7a}\right)^2,\ \left(\frac{1}{3\frac{a}{b}}\right)^3,\ (-1)^4,\ (-1)^5,\ \left(\frac{\frac{1}{2}x}{\frac{1}{4}y}\right)^3,\ \left(\frac{a^{\frac{m}{n}}}{b^{\frac{p}{q}}}\right)^r.$

Ans. $\frac{144a^4b^6}{49a^2},\ \frac{1}{27\frac{a^3}{b^3}},\ 1,\ -1,\ \frac{\frac{1}{8}x^3}{\frac{1}{64}y^3},\ \frac{a^{\frac{rm}{n}}}{b^{\frac{pr}{q}}}.$

4. $\left(\frac{a}{b}\right)^0,\ \left(\frac{a}{b}\right)^{-1},\ \left(-\frac{a}{b}\right)^{2m},\ \left(\frac{a}{b}\right)^{-\frac{m}{n}},\ \left(-\frac{a}{b}\right)^{\frac{1}{2}},\ \left(\frac{a}{b}\right)^{\frac{2}{5}},\ \frac{(3a^{-1}b)^3}{(2a^2b^{\frac{1}{2}})^2}.$

Ans. $1,\ \frac{a^{-1}}{b^{-1}},\ \frac{a^{2m}}{b^{2m}},\ \frac{a^{-\frac{m}{n}}}{a^{-\frac{m}{n}}},\ -\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}},\ \frac{a^{\frac{2}{5}}}{b^{\frac{2}{5}}},\ \frac{27a^{-3}b^3}{4a^4b}.$

37. The Roots of Quantities.

The operation of extracting the root of a monomial is just the reverse of that of raising it to a power. Practically,

Extract the required root of the numerical factor, if there be any, and divide the exponents of the literal factors by the index of the root.

This process is called *Evolution.*

If a quantity be raised to an even power, the sign of the power must be plus, whether the quantity (that is the root) is positive or negative; thus,

$$a^2 = a \times a \text{ or } -a \times -a,$$

$$a^4 = a \times a \times a \times a \text{ or } -a \times -a \times -a \times -a.$$

Whence it follows, that when the index of the required root is even, the root may have either the plus or minus sign; it is therefore given both; thus,

$$\sqrt{a^2} = \pm a; \qquad \sqrt[4]{a^4} = \pm a.$$

$$\sqrt{4} = \pm 2; \qquad \sqrt[4]{16} = \pm 2.$$

$$\sqrt[2m]{a^{2m}} = \pm a.$$

Hence we may say that,

The even root of a quantity has the double sign $\pm$.

When the root is odd, the sign of the power must always be the same as that of the root; thus,

$$-a\times -a\times -a=-a^3;\quad (-a)^5=-a^5;\quad (-a)^{2m+1}=-a^{2m+1}$$
$$a\times a\times a\times a=+a^3;\quad (+a)^5=+a^5;\quad (+a)^{2m+1}=+a^{2m+1}$$

It follows, therefore, that,

An odd root always has the same sign as that of the quantity itself.

Examples.

Extract the roots of the following, as indicated:

1. $\sqrt{a^4}$, $\sqrt{4a^2}$, $\sqrt{\tfrac{1}{4}a^2}$, $\sqrt{9a^4b^2}$, $\sqrt[3]{8a^3}$, $\sqrt[3]{-8a^3}$, $\sqrt[3]{\tfrac{1}{8}a^6b^3}$.

Ans. $\pm a^2$, $\pm 2a$, $\pm\tfrac{1}{2}a$, $\pm 3a^2b$, $2a$, $-2a$, $\tfrac{1}{2}a^2b$.

2. $\sqrt{\dfrac{4a^2b^4}{16c^6d^2}}$, $\sqrt[3]{\dfrac{1}{125a^{-6}}}$, $\sqrt{64a^4b^{2m}}$, $\sqrt[m]{a^m}$, $\sqrt[m+1]{a^{m+1}b^{3(m+1)}}$.

Ans. $\pm\dfrac{2ab^2}{4c^3d}$, $\dfrac{1}{5a^{-2}}$, $\pm 8a^2b^m$, a, ab^3.

3. $\sqrt{6a^3c^4d}$, $\sqrt[m]{2ab^{\frac{1}{2}}c^2}$, $\sqrt{36a^{-1}b^{-m}}$, $\sqrt[n]{\dfrac{a^n}{b}}$.

Ans. $\pm 6^{\frac{1}{2}}a^{\frac{3}{2}}c^2d^{\frac{1}{2}}$, $2^{\frac{1}{m}}a^{\frac{1}{m}}b^{\frac{1}{2m}}c^{\frac{2}{m}}$, $\pm 6a^{-\frac{1}{2}}b^{-\frac{m}{2}}$, $\dfrac{a}{b^{\frac{1}{n}}}$.

4. $\sqrt[3]{-27a^{3n}b^6}$, $\sqrt[3]{27a^{3m}b^6}$, $\sqrt[5]{-\dfrac{a}{b}}$, $\sqrt[5]{\dfrac{a}{b}}$.

Ans. $-3a^nb^2$, $3a^mb^2$, $-\dfrac{a^{\frac{1}{5}}}{b^{\frac{1}{5}}}$, $\left(\dfrac{a}{b}\right)^{\frac{1}{5}}$.

5. $\sqrt{1}$, $\sqrt[3]{-1}$, $\sqrt[3]{1}$, $\sqrt[5]{\dfrac{a^5}{32}}$, $\sqrt[5]{-\dfrac{a^5}{32}}$.

Ans. ± 1, -1, 1, $\dfrac{a}{2}$, $-\dfrac{a}{2}$.

There is no such thing, so far as the human understanding can reach, as the *even root* of a *negative quantity;* for we cannot conceive of a quantity which is not either + or −: but the product of any quantity, + or —, taken an even number of times as a factor, is +; therefore, since an even root must enter the power an even number of times, we cannot conceive of such a negative power; thus, $\sqrt{-a^2}$ can be neither $+a$ nor $-a$; for $+a\times +a=+a^2$ and $-a\times -a=+a^2$: so that $-a^2$ cannot be produced by the multiplication of any thing whatever by itself. Such indicated roots are called *Imaginary*. In general,

An imaginary quantity is the indicated EVEN *root of a negative quantity;* thus,

$$\sqrt{-4},\ \sqrt[4]{-a},\ \sqrt{-3},\ \sqrt[6]{-a^6b^2},\text{ and }\sqrt{-7a}$$

are all imaginary.

38. The Division of Monomials.

Division and *Multiplication* are reciprocal operations. Any quantity united to another by the sign × may be replaced by its reciprocal (unity divided by it) with the sign ÷; thus, $5\times 2=5\div\frac{1}{2}$ or $a\times b=a\div\frac{1}{b}$. Whatever has been said, therefore, of *multiplication*, taken in the converse sense, will apply to *division*.

The signs × *and* ÷ *always indicate operations upon factors.* The sign × means *to add as a factor;* and ÷ means to subtract as a factor. $a^3\times a$ shows that the quantity a^3 is to be further augmented by the introduction of another a as a factor, giving a^4; while $a^3\div a$ shows that a factor a is to be withdrawn from a^3, giving a^2.

To divide one quantity by another, is to withdraw the divisor as a factor from the dividend. Where a factor of the divisor, or the divisor itself, enters the dividend exactly, it may be taken out at once; thus,

$$\frac{15a^2b^4cd}{3ab^3c}=5abd.$$

Numerals are disposed of as in arithmetic, and where the same letter is found in dividend and divisor, the difference of their exponents gives the new exponent of that letter in the quotient.

Let it be remembered that like signs give + and unlike —.

EXAMPLES.

1. Simplify $25a^3bc^2d\div 5a^2c$. *Ans.* $5abcd$.

2. Simplify $9a^5b^3c^2x^2\div 3a^3cx$. *Ans.* $3a^2b^3cx$.

3. Simplify $\frac{a^{m+1}b^3y^2}{-a^mb^2y}$. *Ans.* $-aby$.

4. Simplify $\frac{-a^3b^2x^n}{abx^2}$. *Ans.* $-a^2bx^{n-2}$.

5. Simplify $\frac{-a^mb^ny^p}{-aby}$. *Ans.* $a^{m-1}b^{n-1}y^{p-1}$.

39. The Zero Power of a Quantity.

In dividing one quantity by another it often happens that the exponents of the same letter in the two quantities are the same, in which case that letter disappears entirely; thus, $\frac{ab^2}{b^2} = a$.

We may, however, leave the letter in the quotient by giving it the exponent zero; thus, $\frac{ab^2}{b^2} = ab^\circ$. Here, manifestly, $b^\circ = 1$. The zero simply shows that b is not a factor of the quotient at all. To comprehend how any quantity to the zero power must be equal to unity, we have but to remember that unity enters every expression as a factor, and that thus $a^\circ = 1 . a^\circ$.

Now, removing a, since the exponent $^\circ$ shows that it is not a factor, the co-efficient 1 remains.

EXAMPLES.

1. $\frac{3a^2bc^{\frac{1}{2}}}{a^2b}$, $\frac{25a^mb^{\frac{1}{2}}c,}{-5a^mb}$, $\frac{-x^{\frac{m}{n}}y^{\frac{p}{q}}}{xy}$, $\frac{-1}{-1}$, $\frac{-a}{a}$.

Ans. $3c^{\frac{1}{2}}$, $-5b^{-\frac{1}{2}}c$, $-x^{\frac{m}{n}-1}y^{\frac{p}{q}-1}$, 1, -1.

2. $\frac{a^\circ}{b^\circ}$, $\frac{-a^\circ}{b^\circ}$, $\frac{3ax^\circ}{3a}$, $\frac{\frac{1}{2}x^\circ(y^\circ)}{-(\frac{1}{2})^\circ}$, $\frac{\left(\frac{a}{b}\right)^\circ(-1)}{(25)^\circ}$.

Ans. 1, -1, 1, $\frac{1}{2}$, -1.

3. $\frac{(5)^mx^{\frac{1}{n}}y^3}{(5)^nx^{-1}y^2}$, $\frac{(a+b)^3}{(a+b)^2}$, $\frac{(a-b)^m(a+b)^n}{(a-b)a}$, $\frac{-[a+(a^2b^4)^2]}{-[+(-1)]}$.

Ans. $5^{m-n}x^{\frac{1}{n}+1}y$, $(a+b)$, $\frac{(a-b)^{m-1}(a+b)^n}{a}$, $-a-a^4b^8$.

4. $\frac{a^m}{a^m} = a^{m-m} = a^\circ$. But $\frac{a^m}{a^m} = 1 \therefore a^\circ = 1$.

40. Entire and Negative Exponents.

When the exponent of a factor in the divisor is greater than the exponent of the same letter in the dividend, the subtraction of the greater from the less must give a negative exponent; thus,

$$\frac{3a^2b}{a^4}=3a^{-2}b.$$

Now, in this case, a enters the numerator but twice as a factor, so that when it becomes necessary to take out *four* such factors, we shall have a deficiency of two. This fact the — sign shows: that is to say, we must still withdraw the factor a twice.

This meaning of the negative exponent follows from the general definition. A negative exponent, then, shows that not only is there no factor such as that over which it is written in the quantity, but that such quantity must still be diminished by that factor a certain number of times.

$$a^{-1}=\frac{1}{a};\quad \text{for } \frac{a^2}{a^3}=a^{2-3}=a^{-1}.$$

But $\dfrac{a^2}{a^3}=\dfrac{1}{a}\therefore a^{-1}=\dfrac{1}{a}.$

In general, let p and q be two quantities whose difference is m, q being the greater; then,

$$\frac{a^p}{a^q}=a^{p-q}=a^{-m}.$$

But $\dfrac{a^p}{a^q}=\dfrac{1}{a^m}\therefore a^{-m}=\dfrac{1}{a^m}.$ That is,

Any quantity raised to a negative power is equal to unity divided by the quantity raised to the corresponding positive power.

Again,

$\dfrac{a^3}{a^2}=\dfrac{1}{a^{2-3}}=\dfrac{1}{a^{-1}}$; but

$\dfrac{a^3}{a^2}=a\therefore a=\dfrac{1}{a^{-1}}$; or, in general, let p and q be two quantities whose difference is m, q being the greater.

$$\frac{a^q}{a^p}=\frac{1}{a^{p-q}}=\frac{1}{a^{-m}};\ \text{but}$$

$$\frac{a^q}{a^p}=a^m\ \therefore\ a^m=\frac{1}{a^{-m}}.\quad \text{That is,}$$

Any quantity raised to a positive power is equal to unity divided by that quantity raised to a corresponding negative power.

It follows from the foregoing that,

Any factor may be transferred from the numerator to the denominator; or from the denominator to the numerator by changing the sign of its exponent.

EXAMPLES.

Convert the following expressions into fractional forms with unity for numerators:

1. a, ab, $a^{\frac{1}{2}}$, a^m, $a^{\frac{1}{n}}b^3$, $5ab^{\frac{m}{n}}$, $20x^{-1}y^2$.

Ans. $\frac{1}{a^{-1}}$, $\frac{1}{a^{-1}b^{-1}}$, $\frac{1}{a^{-\frac{1}{2}}}$, $\frac{1}{a^{-m}}$, $\frac{1}{a^{-\frac{1}{n}}b^{-3}}$, $\frac{1}{5^{-1}a^{-1}b^{-\frac{m}{n}}}$, $\frac{1}{20^{-1}xy^{-2}}$.

2. $(a+b)^{-1}$, $\left(\frac{a}{b}\right)^2$, $a^{-1}b^m$, $x^{-\frac{m}{n}}y^p$. $-a$, $-(-a)^{-1}$.

Ans. $\frac{1}{(a+b)}$, $\frac{1}{\left(\frac{a}{b}\right)^{-2}}$, $\frac{1}{ab^{-m}}$, $\frac{1}{x^{\frac{m}{n}}y^{-p}}$, $\frac{-1}{a^{-1}}$, $\frac{-1}{-a}=\frac{1}{a}$.

Convert the following into expressions containing no negative exponents.

3. $8a^{-1}$, $\frac{a^{-m}b}{b^{-1}c}$, $\frac{3^{-1}xy^{-\frac{1}{2}}}{2^{-1}ab^{-m}}$, $\frac{(a+b)^{-1}}{(a-b)^{-1}}$, $\frac{(a-b)(a+b)^{-1}}{(a-b)^{-1}(a+b)}$.

Ans. $\frac{8}{a}$, $\frac{b^2}{a^mc}$, $\frac{2xb^m}{3ay^{\frac{1}{2}}}$, $\frac{a-b}{a+b}$, $\frac{(a-b)(a-b)}{(a+b)(a+b)}$.

Convert numerators into denominators, and the converse.

4. $\frac{7ab^{-1}}{2c^{-1}d}$, $\frac{a^{-\frac{1}{2}}b^m}{cd^{-p}}$, $\frac{\frac{2}{3}a^bcx^{-\frac{1}{3}}}{d^mc^q}$, $\frac{9^{-1}xy}{-1}$.

Ans. $\frac{2^{-1}cd^{-1}}{7^{-1}a^{-1}b}$, $\frac{c^{-1}d^p}{a^{\frac{1}{2}}b^{-m}}$, $\frac{d^{-m}c^{-q}}{(\frac{2}{3})^{-1}a^{-b}c^{-1}x^{\frac{1}{3}}}$, $\frac{(-1)^{-1}}{9x^{-1}y^{-1}}$.

Remark.—It must be carefully borne in mind that only factors can be thus transferred. Thus, in the expression $\frac{a+b}{c}$, the a or b cannot be written in the denominator. The $a+b$ must be taken together, thus, $\frac{1}{c(a+b)^{-1}}$.

We could write, however,

$$\frac{\frac{1}{a^{-1}}+b}{c} \text{ or } \frac{\frac{1}{a^{-1}}+\frac{1}{b^{-1}}}{c}.$$

5. $$\frac{a-b}{a+c}=\frac{(a+c)^{-1}}{(a-b)^{-1}}, \text{ or } \frac{\frac{1}{a^{-1}}-\frac{1}{b^{-1}}}{\frac{1}{a^{-1}}+\frac{1}{c^{-1}}}, \text{ or } (a-b)(a+c)^{-1}.$$

6. $$\frac{\frac{a}{b}-\frac{a}{c}}{d}=\frac{ab^{-1}-ac^{-1}}{d}=(ab^{-1}-ac^{-1})d^{-1}.$$

41. Fractional Exponents.– Radicals.

From the general definition of an exponent, it follows that a fractional exponent must show how many times the quantity to which it is affixed is to be reckoned as a factor: that is to say, that the quantity is to be resolved into as many equal factors as there are units in the denominator of the exponent, and that as many of these factors are to be taken, as there are units in the numerator; thus

$$(9)^{\frac{1}{2}}=(3\times 3)^{\frac{1}{2}}=3.$$

$$a^{\frac{1}{2}}=(\sqrt{a}\cdot\sqrt{a})^{\frac{1}{2}}=\sqrt{a}.$$

$$a^{\frac{1}{m}}=(\sqrt[m]{a}\cdot\sqrt[m]{a}\cdot\sqrt[m]{a}\cdot\cdots\cdots m \text{ times})^{\frac{1}{m}}=\sqrt[m]{a}.$$

Now, the *Root* of a quantity is that factor of it which, taken a certain number of times, will produce the quantity itself.

If there are but two equal factors, either of them is the *square root;* if there are three, any one of them is the *cube root;* if there are m equal factors, any one of them is the mth root; thus,

$25=5\times 5 \quad \therefore\ 5=\sqrt{25}$; $\quad 27=3\times 3\times 3 \quad \therefore\ 3=\sqrt[3]{27}$. In like manner, if a contains b m times as a factor, $b=\sqrt[m]{a}$.

It follows therefore, that,

The denominator of a fractional exponent shows the root to be extracted; thus,

$$a^{\frac{1}{2}}=\sqrt{a}$$

$$a^{\frac{1}{3}}=\sqrt[3]{a}$$

$$a^{\frac{1}{m}}=\sqrt[m]{a}.$$

It is thus manifest, that,

The radical sign is identical in signification with a fractional exponent, having unity for its numerator and the index of the radical for its denominator. Any radical may, therefore, be removed by enclosing the quantity in a parenthesis, and writing instead of it such a fractional exponent.

There is thus no necessity for the use of the radical, and it is only retained here because it is so universally found in mathematical works.

EXAMPLES.

Transform the following into expressions with fractional exponents:

1. $\sqrt{5}$, $\sqrt[3]{c}$, $\sqrt[n]{d}$, $\sqrt{a+b}$. *Ans.* $5^{\frac{1}{2}}$, $c^{\frac{1}{3}}$, $d^{\frac{1}{n}}$, $(a+b)^{\frac{1}{2}}$.

Remark.—When the radical sign is removed from a polynomial, as in the last example, the bar must be retained, thus, $\overline{a+b}^{\frac{1}{2}}$, or replaced by a parenthesis, thus, $(a+b)^{\frac{1}{2}}$; so, also, where any mistake might result, as in $\sqrt{\frac{2}{3}}$, or $\sqrt[3]{\frac{a}{b}}$, the parenthesis must be used; thus, $(\frac{2}{3})^{\frac{1}{2}}$ and $\left(\frac{a}{b}\right)^{\frac{1}{3}}$.

2. $\sqrt{\dfrac{a}{b}}=\left(\dfrac{a}{b}\right)^{\frac{1}{2}}$, $\sqrt{\dfrac{a}{a-c}}=\left(\dfrac{a}{a-c}\right)^{\frac{1}{2}}=\dfrac{a^{\frac{1}{2}}}{(a-c)^{\frac{1}{2}}}$.

3. $\sqrt{\dfrac{25x^m}{3y^{-1}}}=\dfrac{(25)^{\frac{1}{2}}(x^m)^{\frac{1}{2}}}{3^{\frac{1}{2}}(y^{-1})^{\frac{1}{2}}}$, $\sqrt[3]{\dfrac{a^2b^2}{c(a-b)}}=\dfrac{a^{\frac{2}{3}}b^{\frac{2}{3}}}{c^{\frac{1}{3}}(a-b)^{\frac{1}{3}}}$.

4. $\sqrt[3]{\dfrac{1}{x}}=\dfrac{1}{x^{\frac{1}{3}}}$, $\sqrt[n]{y^m}=y^{\frac{m}{n}}$, $\sqrt[4]{\dfrac{1}{x^3}}=\dfrac{1}{(x^3)^{\frac{1}{4}}}=x^{-\frac{3}{4}}$, $\sqrt[n]{\dfrac{1}{x^m}}=\dfrac{1}{(x^m)^{\frac{1}{n}}}=x^{-\frac{m}{n}}$.

5. $\sqrt{a}\times\sqrt{b}=a^{\frac{1}{2}}b^{\frac{1}{2}}=(ab)^{\frac{1}{2}}$, $\sqrt[n]{a}\cdot\sqrt[m]{b}=a^{\frac{1}{n}}b^{\frac{1}{m}}$.

6. $\sqrt{\sqrt{a}}=\left(\sqrt{a}\right)^{\frac{1}{2}}=\left(a^{\frac{1}{2}}\right)^{\frac{1}{2}}=a^{\frac{1}{4}}$, $\sqrt[m]{\sqrt[n]{a}}=\left(\sqrt[n]{a}\right)^{\frac{1}{m}}=\left(a^{\frac{1}{n}}\right)^{\frac{1}{m}}=a^{\frac{1}{mn}}$.

7. $a^{\frac{1}{n}}=\sqrt[n]{a}$, $a^{\frac{2}{3}}b^{\frac{3}{4}}=\sqrt[3]{a^2}\cdot\sqrt[4]{b^3}$, $a^{\frac{1}{m}}b^{\frac{1}{q}}c^{-1}=\sqrt[m]{a}.\sqrt[q]{b}\left(\dfrac{1}{c}\right)$.

42. Fractional Exponents.—Radicals.

Now, while the denominator of a fractional exponent shows the degree of the root to be extracted, the numerator shows the number of times such root is to be taken; that is, it shows the *power* to which this root is to be raised; thus,

$$8^{\frac{2}{3}}=2^2=4.$$
$$a^{\frac{3}{5}}=\left(\sqrt[5]{a}\right)^3.$$
$$a^{\frac{m}{n}}=\left(\sqrt[n]{a}\right)^m.$$

But, $(\sqrt[n]{a})^m=\sqrt[n]{a^m}$, for we may write $(\sqrt[n]{a})^m=a^{\frac{m}{n}}=(a^m)^{\frac{1}{n}}=\sqrt[n]{a^m}$; hence, we can say in general, that

The numerator of a fractional exponent shows the power to which the quantity affected by it is to be raised, while the denominator shows the root to be extracted.

Examples.

Transform the following into equivalent expressions:

1. $\sqrt[m]{a^n},\quad a^{\frac{p}{q}},\quad a^{-\frac{m}{n}},\quad a^{\frac{1}{2}m},\quad a^{\frac{pq}{n}},\quad a^{\frac{1}{m+n}}.$

Ans. $a^{\frac{n}{m}},\quad \sqrt[q]{a^p},\quad \sqrt[n]{\frac{1}{a^m}},\quad \sqrt{a^m},\quad \sqrt[n]{a^{pq}},\quad \sqrt[m+n]{a}.$

2. $\sqrt[m]{\sqrt[n]{a^2}},\quad \sqrt[m]{a^2}\cdot\sqrt[n]{a^3}\cdot\sqrt[p]{a^4},\quad \sqrt{x^{\frac{1}{2}}}\cdot\sqrt{y^{\frac{1}{3}}}.$

Ans. $\left(\sqrt[n]{a^2}\right)^{\frac{1}{m}}=\left(a^{\frac{2}{n}}\right)^{\frac{1}{m}},\quad a^{\frac{2}{m}}a^{\frac{3}{n}}a^{\frac{4}{p}},\quad \left(x^{\frac{1}{2}}\right)^{\frac{1}{2}}\left(y^{\frac{1}{3}}\right)^{\frac{1}{2}}=x^{\frac{1}{4}}y^{\frac{1}{6}}.$

3. $\sqrt{28}\cdot\sqrt[3]{5}\cdot\sqrt[4]{a},\quad \sqrt{\frac{a}{b}}\cdot\sqrt{\frac{c}{d}}\cdot\sqrt{\frac{e}{f}}.$

Ans. $28^{\frac{1}{2}}\cdot5^{\frac{1}{3}}\cdot a^{\frac{1}{4}},\quad \left(\frac{a}{b}\right)^{\frac{1}{2}}\left(\frac{c}{d}\right)^{\frac{1}{2}}\left(\frac{e}{f}\right)^{\frac{1}{2}}=\left(\frac{ace}{bdf}\right)^{\frac{1}{2}}.$

4. $\frac{a\sqrt{b}}{c\sqrt{d}},\quad \frac{5\sqrt[n]{a^2}}{6\sqrt[m]{a^3}},\quad \frac{3a^2b(cd)^{\frac{1}{m}}}{2a^{-m}},\quad \frac{cd(25)^{\frac{1}{2}}}{4ax}.$

Ans. $\frac{ab^{\frac{1}{2}}}{cd^{\frac{1}{2}}},\quad \frac{5a^{\frac{2}{n}}}{6a^{\frac{3}{m}}};\quad \frac{3a^2b\sqrt[m]{cd}}{\frac{2}{a^m}},\quad \frac{5cd}{4ax}.$

43. To change the Index of a Radical.

It is obvious that we can multiply or divide the numerator and denominator of a fractional exponent by the same thing, without changing its signification; thus, $a^{\frac{2}{3}}=a^{\frac{4}{6}}$, $a^{\frac{m}{n}}=a^{\frac{2m}{2n}}$, $a^{\frac{pm}{pn}}=a^{\frac{m}{n}}$. But the numerator is the exponent of the power and the denominator is the index of the root; whence we may say, in general, that

We can multiply or divide the index of a radical by any quantity, provided that we, at the same time, multiply or divide the exponent of the quantity under the radical sign.

For example, take $\sqrt[3]{a^2+b-c}$. If we should multiply the index of the root by 2, we must raise a^2+b-c to the second power; thus, $\sqrt[3]{a^2+b-c}=\sqrt[6]{(a^2+b-c)^2}$. The quantity under the radical must be considered as one quantity. We may, however, multiply or divide the exponent of each factor under the sign; thus, $\sqrt{3a^2b}=\sqrt[4]{9a^4b^2}$, or $\sqrt[6]{36a^4b^6}=\sqrt[3]{6a^2b^3}$.

Examples.

Change the indices of the following by multiplying by 2.

1. $\sqrt{a}$, $\sqrt{5}$, $\sqrt[3]{x}$, $\sqrt{3a^2b}$, $\sqrt[m]{3a^2b}$, $\sqrt[n]{3a^mb^n}$.

Ans. $\sqrt[4]{a^2}$, $\sqrt[4]{25}$, $\sqrt[6]{x^2}$, $\sqrt[4]{9a^4b^2}$, $\sqrt[2m]{9a^4b^2}$, $\sqrt[2n]{9a^{2m}b^{2n}}$.

Multiply by 3.

2. $\dfrac{\sqrt{a}}{\sqrt{b}}$, $\dfrac{\sqrt[3]{a+b}}{c}$, $\dfrac{\sqrt[m]{15a^2x^{-1}}}{\sqrt[n]{a^{\frac{1}{2}}}}$, $\sqrt[5]{\dfrac{a^{-\frac{1}{2}}}{b^m}}$, $\dfrac{\sqrt{a-b}}{\sqrt[3]{a-b}}$.

Ans. $\dfrac{\sqrt[6]{a^3}}{\sqrt[6]{b^3}}$, $\dfrac{\sqrt[9]{(a+b)^3}}{c}$, $\dfrac{\sqrt[3m]{(15)^3a^6x^{-3}}}{\sqrt[3n]{a^{\frac{3}{2}}}}$, $\sqrt[15]{\dfrac{a^{-\frac{3}{2}}}{b^{3m}}}$, $\dfrac{\sqrt[6]{(a-b)^3}}{\sqrt[9]{(a-b)^3}}$.

Multiply by m.

3. $\sqrt{a^2b}$, $\sqrt{\dfrac{3a}{2b}}$, $\sqrt[3]{\dfrac{7ax^{-1}}{b^{\frac{1}{2}}}}$, $\sqrt{\dfrac{a^{\frac{2}{3}}b^{\frac{m}{n}}}{c^{-1}d^{-\frac{1}{2}}}}$.

Ans. $\sqrt[2m]{a^{2m}b^m}$, $\sqrt[2m]{\dfrac{3^ma^m}{2^mb^m}}$, $\sqrt[3m]{\dfrac{7^ma^mx^{-m}}{b^{\frac{m}{2}}}}$, $\sqrt[2m]{\dfrac{a^{\frac{2m}{3}}b^{\frac{m^2}{n}}}{c^{-m}d^{-\frac{m}{2}}}}$.

Divide by 2.

4. $\sqrt[4]{a^2}$, $\sqrt[6]{4a^4b^2}$, $\sqrt{64x^2y^{-4}}$, $\sqrt[4]{\dfrac{36a^{-2}b}{25xy^4}}$.

Ans. $\sqrt{a}$, $\sqrt[3]{2a^2b}$, $8xy^{-2}$, $\sqrt{\dfrac{6a^{-1}b^{\frac{1}{2}}}{5x^{\frac{1}{2}}y^2}}$.

44. To bring Radicals to a common Index.

Radical quantities with different indices may be changed into equivalent radicals having the same indices. For let $\sqrt[m]{a^p}$ and $\sqrt[n]{b^q}$ be two such quantities. They may be written,

$$a^{\frac{p}{m}} \text{ and } b^{\frac{q}{n}}.$$

Causing these fractional exponents to have a common denominator, we have,

$$a^{\frac{pn}{mn}} \text{ and } b^{\frac{qm}{mn}};$$

and thus, $\sqrt[mn]{a^{pn}}$ and $\sqrt[mn]{b^{qm}}$. We may, therefore, say that,

To bring radicals to a common index, find a common multiple of the several indices, and take this for the common index; then divide this new index by each of the old ones in succession, and raise each quantity under the radicals, respectively, to the power indicated by these quotients; or

Use fractional exponents and bring them to a common fractional unit.

For example, take $4\sqrt{2a}$, $\frac{1}{2}\sqrt[3]{3b^2}$ and $a\sqrt[4]{c^3}$. The least common multiple of the indices is 12. We may then write $4\sqrt[12]{\quad}$, $\frac{1}{2}\sqrt[12]{\quad}$, $a\sqrt[12]{\quad}$; now raising the quantities under the radicals, respectively, to the powers found by dividing 12 by 2, by 3 and by 4, we have,

$$4\sqrt[12]{(2a)^6},\quad \tfrac{1}{2}\sqrt[12]{(3b^2)^4} \text{ and } a\sqrt[12]{(c^3)^3}; \text{ or}$$
$$4\sqrt[12]{64a^6},\quad \tfrac{1}{2}\sqrt[12]{81b^8} \text{ and } a\sqrt[12]{c^9}.$$

It will be observed that the quantities outside of the radicals remain unchanged.

EXAMPLES.

Transform the following radicals into equivalent ones having common indices:

1. $\frac{1}{2}\sqrt{5a^2}$, $\sqrt[3]{3bc}$, $\frac{1}{a}\sqrt{c}$. *Ans.* $\frac{1}{2}\sqrt[6]{125a^6}$, $\sqrt[6]{9b^2c^2}$, $\frac{1}{a}\sqrt[6]{c^3}$.

2. $a\sqrt[n]{b}$, $\frac{c}{d}\sqrt[m]{3a}$, $\sqrt[p]{\frac{a}{b}}$. *Ans.* $a\sqrt[mnp]{b^{mp}}$, $\frac{c}{d}\sqrt[mnp]{3^{np}a^{np}}$, $\sqrt[mnp]{\frac{a^{mn}}{b^{mn}}}$.

3. $\sqrt{a+b}$, $\sqrt[3]{\frac{a}{a-b}}$. *Ans.* $\sqrt[6]{(a+b)^3}$, $\sqrt[6]{\frac{a^2}{(a-b)^2}}$.

4. $a^{\frac{1}{2}}$, $b^{\frac{2}{3}}$. *Ans.* $a^{\frac{3}{6}}$, $b^{\frac{4}{6}}$. $a^{\frac{m}{n}}$, $b^{\frac{p}{q}}$. *Ans.* $a^{\frac{qm}{qn}}$, $b^{\frac{pn}{qn}}$.

5. $(a+b)^{\frac{1}{2}}$, $(a-b)^{\frac{1}{3}}$. *Ans.* $(a+b)^{\frac{3}{6}}$, $(a-b)^{\frac{2}{6}}$.

45. Product of the nth Roots.

The product of the n*th roots of two quantities is equal to the* n*th root of the product of the quantities, and the converse.*

For,

$$a^{\frac{1}{n}}b^{\frac{1}{n}}=(ab)^{\frac{1}{n}};\ \text{whence,}$$

$$\sqrt[n]{a}\cdot\sqrt[n]{b}=\sqrt[n]{ab}.$$

Again,

The quotient of the n*th roots is equal to the* n*th root of the quotient, and the converse,*

For,

$$\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}=\left(\frac{a}{b}\right)^{\frac{1}{n}};\ \text{whence,}$$

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}.$$

These principles enable us to combine radicals by multiplication or division. Thus,

$$a\sqrt{b}\times c\sqrt{d}=ac\sqrt{b}\cdot\sqrt{d}=ac\sqrt{bd};$$

hence,

To multiply radical quantities together, multiply the factors outside of the radical signs together, and also the factors under the radical signs, retaining the radical over the latter product.

For example,

$$3a^2b\sqrt{5cd^m}\times 2ab^3\sqrt{cd}=3a^2b\times 2ab^3\sqrt{5cd^m\times cd}=6a^3b^4\sqrt{5c^2d^{m+1}}.$$

The indices of radical expressions so combined must be the same; if not so already, they must be changed into corresponding radicals with a common index.

EXAMPLES.

1. $3a\sqrt{b^2} \times 5b\sqrt{7}$. *Ans.* $15ab\sqrt{7b^2}$.

2. $3a(b^2)^{\frac{1}{2}} \times 5b(7)^{\frac{1}{2}}$. *Ans.* $15ab(7b^2)^{\frac{1}{2}}$.

3. $\frac{1}{2}\sqrt{a} \times \frac{1}{3}\sqrt{b} \times -\sqrt{\frac{1}{a}}$. *Ans.* $-\frac{1}{6}\sqrt{b}$.

4. $\frac{1}{2}(a)^{\frac{1}{2}} \times \frac{1}{3}(b)^{\frac{1}{2}} \times -\left(\frac{1}{a}\right)^{\frac{1}{2}}$. *Ans.* $-\frac{1}{6}(b)^{\frac{1}{2}}$.

5. $\frac{3a^2b}{x}\sqrt[3]{4c} \times \frac{1}{2}\sqrt[3]{\frac{1}{3}}$. *Ans.* $\frac{3a^2b}{2x}\sqrt[3]{\frac{4c}{3}}$.

6. $\frac{3a^2b}{x}(4c)^{\frac{1}{3}} \times \frac{1}{2}(\frac{1}{3})^{\frac{1}{3}}$. *Ans.* $\frac{3a^2b}{2x}\left(\frac{4c}{3}\right)^{\frac{1}{3}}$.

7. $-\sqrt{2a} \times \sqrt[3]{2a} \times \frac{1}{2}\sqrt[4]{2a}$. *Ans.* $-\frac{1}{2}\sqrt[12]{8192a^{13}}$.

8. $\sqrt[m]{a^n} \times -\sqrt[n]{a^m} \times -\sqrt[p]{a}$. *Ans.* $\sqrt[mnp]{a^{n^2p+m^2p+mn}}$.

9. $a^{\frac{n}{m}} \times -a^{\frac{m}{n}} \times -a^{\frac{1}{p}}$. *Ans.* $a^{\frac{n2p}{mnp}+\frac{m2p}{mnp}+\frac{mn}{mnp}}$.

10. $2\sqrt{\frac{a}{b}} \times -c\sqrt{\frac{c}{d}}$. *Ans.* $-2c\sqrt{\frac{ac}{bd}}$.

11. $2\left(\frac{a}{b}\right)^{\frac{1}{2}} \times -c\left(\frac{c}{d}\right)^{\frac{1}{2}}$. *Ans.* $-2c\left(\frac{ac}{bd}\right)^{\frac{1}{2}}$.

12. $5a\sqrt{-3x} \times -2b\sqrt[3]{2x^2}$. *Ans.* $-10ab\sqrt[6]{-108x^7}$.

13. $5a(-3x)^{\frac{1}{2}} \times -2b(2x^2)^{\frac{1}{3}}$.

Ans. $-10ab(-27x^3)^{\frac{1}{6}}(4x^4)^{\frac{1}{6}} = -10ab(-108x^7)^{\frac{1}{6}}$.

46. Division of Radicals.

To show how to divide one radical expression by another, we have,

$$\frac{a\sqrt[m]{b}}{c\sqrt[m]{d}} = \frac{a}{c} \cdot \frac{\sqrt[m]{b}}{\sqrt[m]{d}} = \frac{a}{c}\sqrt[m]{\frac{b}{d}}; \text{ whence,}$$

To divide one radical quantity by another, divide the quantities *without the signs, and the quantities within, respectively, retaining the radical sign over the latter quotient.*

The indices must be made common, if not so already.

EXAMPLES.

1. $\dfrac{\sqrt{5}}{\sqrt{4}}=\sqrt{\dfrac{5}{4}}$, $\dfrac{3\sqrt{a}}{2\sqrt{b}}=\dfrac{3}{2}\sqrt{\dfrac{a}{b}}$, $\dfrac{3a^2\sqrt{\frac{1}{2}}}{-\frac{1}{2}\sqrt{a}}=-6a^2\sqrt{\dfrac{1}{2a}}$,

$$\frac{\frac{1}{2}\sqrt{\frac{1}{2}}}{\frac{1}{3}\sqrt{\frac{a}{b}}}=\frac{3}{2}\sqrt{\frac{b}{2a}}.$$

2. $\dfrac{(5)^{\frac{1}{2}}}{(4)^{\frac{1}{2}}}=\left(\dfrac{5}{4}\right)^{\frac{1}{2}}$, $\dfrac{3(a)^{\frac{1}{2}}}{2(b)^{\frac{1}{2}}}=\dfrac{3}{2}\left(\dfrac{a}{b}\right)^{\frac{1}{2}}$, $\dfrac{3a^2(\frac{1}{2})^{\frac{1}{2}}}{-\frac{1}{2}(a)^{\frac{1}{2}}}=-6a^2\left(\dfrac{1}{2a}\right)^{\frac{1}{2}}$,

$$\frac{\frac{1}{2}\left(\frac{1}{2}\right)^{\frac{1}{2}}}{\frac{1}{3}\left(\frac{a}{b}\right)^{\frac{1}{2}}}=\frac{3}{2}\left(\frac{b}{2a}\right)^{\frac{1}{2}}.$$

3. $\dfrac{b(a)^{\frac{1}{3}}}{c(b)^{\frac{1}{2}}}=\dfrac{b}{c}\left(\dfrac{a^2}{b^3}\right)^{\frac{1}{6}}$, $\dfrac{\left(\frac{a}{b}\right)^{\frac{1}{2}}}{\left(\frac{a}{b}\right)^{\frac{1}{3}}}=\left(\dfrac{a}{b}\right)^{\frac{1}{6}}$, $\dfrac{-(-1)^{\frac{1}{3}}}{(a^{-1})^{\frac{1}{2}}}=(a^3)^{\frac{1}{6}}=a^{\frac{1}{2}}$,

$$\frac{(2a^{-1})^{\frac{1}{n}}}{(c^{-2})^{\frac{1}{m}}}=\left(\frac{2^m c^{2n}}{a^m}\right)^{\frac{1}{mn}}.$$

4. $\dfrac{b\sqrt[3]{a}}{c\sqrt{b}}=\dfrac{b}{c}\sqrt[6]{\dfrac{a^2}{b^3}}$, $\dfrac{\sqrt{\frac{a}{b}}}{\sqrt[3]{\frac{a}{b}}}=\sqrt[6]{\dfrac{a}{b}}$, $\dfrac{-\sqrt[3]{\frac{1}{2}}}{\sqrt{a^{-1}}}=-\sqrt[6]{\dfrac{a^3}{4}}$, $\dfrac{\sqrt[n]{2a^{-1}}}{\sqrt[m]{c^{-2}}}$

$$=\sqrt[mn]{\frac{2^m c^{2n}}{a^m}}.$$

5. $\dfrac{3x(2a^2)^{\frac{2}{3}}}{2y(a^{-1})^{\frac{4}{5}}}=\dfrac{3x}{2y}\left(\dfrac{2^{10}a^{20}}{a^{-12}}\right)^{\frac{1}{15}}$, $\dfrac{8a^2b(5ab^2)^{\frac{m}{n}}}{2ab(7cb^{-1})^{\frac{1}{2}}}=4a\left(\dfrac{5^{2m}a^{2m}b^{4m}}{7^n c^n b^{-n}}\right)^{\frac{1}{2n}}$.

47. Simplification of Radicals.

Any factor may be removed from within a parenthesis and made a factor without, by multiplying its exponent by the exponent of the parenthesis; thus,

$$2a(b^2c^{3m}d)^2 = 2ab^4(c^{3m}d)^2.$$

$$2a(b^2c^{3m}d)^{\frac{1}{2}} = 2ab(c^{3m}d)^{\frac{1}{2}}.$$

$$2a(b^2c^{3m}d)^{\frac{1}{3}} = 2ac^m(b^2d)^{\frac{1}{3}}.$$

$$2a(b^2c^{3m}d)^{\frac{1}{m}} = 2ab^{\frac{2}{m}}c^3d^{\frac{1}{m}}.$$

Numerical factors may be treated in the same way; thus,

$$2a(12b^2c^3d)^{\frac{1}{2}} = 2a(2^2 \times 3b^2c^2cd)^{\frac{1}{2}}$$

$$= 2 \times 2abc(3cd)^{\frac{1}{2}}$$

$$= 4abc(3cd)^{\frac{1}{2}}.$$

The radical sign means that the root of the quantity under it is to be extracted; which may be done by extracting the root of the several factors successively: whence it follows that we may extract the root of any factor under the radical, and write the root so found as a factor without; thus,

$$2a\sqrt{b^2c^{3m}d} = 2a\sqrt{b^2c^{2m}c^md},$$

$$= 2abc^m\sqrt{c^md},$$

$$2a\sqrt{12b^2c^3d} = 2a\sqrt{4 \times 3b^2c^2cd},$$

$$= 4abc\sqrt{3cd}.$$

This operation is called *simplifying* a radical. A radical is said to have its *simplest form* when there is no factor under the radical of as great a power as the degree of the radical. Practically,

To simplify a radical expression, look out all the factors under the radical sign which are exact powers of the degree of the root required, or may be made so; divide the exponents of such factors by the index, and write the factors, with their new exponents, without the radical.

Or if the parenthesis is used instead of the radical sign, *Multiply the exponent of the factor by the exponent of the parenthesis.*

Examples.

1. $2a(4b^2c)^{\frac{1}{2}} = 2a \times 2b(c)^{\frac{1}{2}} = 4ab(c)^{\frac{1}{2}}$.

2. $2a\sqrt{4b^2c} = 2a \times 2b\sqrt{c} = 4ab\sqrt{c}$.

3. $\frac{1}{2}\left(\frac{a^2b}{c^2}\right)^{\frac{1}{2}} = \frac{1}{2}\left(\frac{a^2}{c^2} \times b\right)^{\frac{1}{2}} = \frac{1}{2} \cdot \frac{a}{c}(b)^{\frac{1}{2}}$.

4. $\frac{1}{2}\sqrt{\frac{a^2b}{c^2}} = \frac{1}{2}\sqrt{\frac{a^2}{c^2} \cdot b} = \frac{a}{2c}\sqrt{b}$.

5. $5x\left(x^2(a+b)\right)^{\frac{1}{2}} = 5x^2(a+b)^{\frac{1}{2}}$.

6. $5x\sqrt{x^2(a+b)} = 5x^2\sqrt{a+b}$.

7. $3a\left(9x^4y(a+b)^2\right)^{\frac{1}{2}} = 9ax^2(a+b)y^{\frac{1}{2}}$.

8. $7a\sqrt{16xy^2(a+b)^4} = 28ay\,(a+b)^2\sqrt{x}$.

9. $c(18a^3b^5)^{\frac{1}{2}} = c(2 \times 9a^2ab^4b)^{\frac{1}{2}} = 3ab^2c(2ab)^{\frac{1}{2}}$.

10. $\frac{1}{2}\sqrt{48a^{m+2}} = \frac{1}{2}\sqrt{3 \times 16a^m a^2} = 2a\sqrt{3a^m}$.

11. $[32a^7(2+b)^4]^{\frac{1}{3}} = [4 \times 8a^6a(2+b)^3(2+b)]^{\frac{1}{3}} = 2a^2(2+b)[4a(2+b)]^{\frac{1}{3}}$.

12. $\sqrt{112a^3} = 4a\sqrt{7a}$, $\sqrt{\frac{50}{8}.a^{m+3}} = \frac{5}{2}a\sqrt{a^{m+1}}$, $\sqrt{\frac{324}{27a^3}} = \frac{6}{3a}\sqrt{\frac{3}{a}}$.

13. $\sqrt[3]{\frac{200}{675}a^7b} = \frac{2}{3}a^2\sqrt{ab}$, $\sqrt{\frac{8}{18}a^{-3}} = \frac{4}{6}a^{-1}\sqrt{a^{-1}}$, $\sqrt{\frac{10}{40}a^{\frac{4}{5}}} = \frac{10}{20}a^2\sqrt{a^{\frac{1}{5}}}$.

14. $\sqrt[3]{\frac{27a^4x^5}{8cd^3}} = \frac{3ax}{2d}\sqrt{\frac{ax^2}{c}}$, $\frac{a}{b}\sqrt[3]{125a^{-3}b^{-6}} = \frac{5b^{-2}}{b} = \frac{5}{b^3}$.

15. $\sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$, $\sqrt{\frac{8}{27}} = \frac{2}{3}\sqrt{\frac{2}{3}}$, $\frac{1}{2}\sqrt{\frac{4}{32}} = \frac{1}{2}\sqrt{\frac{1}{8}}$.

16. $\sqrt[3]{\frac{135}{32}} = \sqrt[3]{\frac{270}{64}} = \sqrt[3]{\frac{27}{64} \times 10} = \frac{3}{4}\sqrt[3]{10}$.

17. $\sqrt[3]{\frac{21}{9}} = \sqrt[3]{\frac{63}{27}} = \sqrt[3]{\frac{1}{27} \times 63} = \frac{1}{3}\sqrt[3]{63}$.

18. $\sqrt{\frac{63}{75}} = \sqrt{\frac{9}{25} \times \frac{7}{3}} = \frac{3}{5}\sqrt{\frac{7}{3}}$.

48. To pass a Factor under the Radical.

It is evident that we may pass a factor from without to within a parenthesis, or the radical sign by the reverse process; that is, by *dividing* the exponent of the factor by the exponent of the parenthesis, in the one case; and by *multiplying* the exponent of the factor by the index of the radical, in the other.

EXAMPLES.

1. $a(a)^{\frac{1}{2}}=(a^3)^{\frac{1}{2}}$, $3a(6a)^{\frac{1}{2}}=(54a^3)^{\frac{1}{2}}$.

2. $2a\sqrt{ab}=2\sqrt{a^3b}$, $3c\sqrt[3]{5a^4d}=\sqrt[3]{135a^4c^3d}$.

3. $2a(a+b)^3=2(a^{\frac{1}{3}}(a+b))^3$, $8a^3(a-b)^3=(2a(a-b))^3$.

4. $5a^2\sqrt{a+b}=5\sqrt{a^4(a+b)}$, $5a\sqrt{a-b}=a\sqrt{25(a-b)}$.

5. $\frac{a}{b}\left(\frac{c}{d}\right)^{\frac{1}{2}}=\left(\frac{a^2}{b^2}\cdot\frac{c}{d}\right)^{\frac{1}{2}}$, $\frac{3a}{2c}\left(\frac{a+b}{c}\right)^{\frac{1}{3}}=\left(\frac{27a^3\,(a+b)}{8c^4}\right)^{\frac{1}{3}}$.

6. $3a^2(a^2b^mc^{-1})^2=(3^{\frac{1}{2}}a(a^2b^mc^{-1}))^2$, $2a^mb^n\left(\frac{a}{b}\right)^{\frac{1}{m}}=\left(\frac{2^ma^{m^2+1}b^{mn}}{b}\right)^{\frac{1}{m}}$

7. $\frac{a}{b}\sqrt[m]{\frac{a}{b}}=\sqrt[m]{\frac{a^{m+1}}{b^{m+1}}}$, $a^{\frac{2}{3}}\sqrt{b^{\frac{1}{2}}}=\sqrt{a^{\frac{4}{3}}b^{\frac{1}{2}}}$.

8. $a^{\frac{m}{n}}(a)^{\frac{m}{n}}=(a^2)^{\frac{m}{n}}$, $a^{\frac{m}{n}}(a)^{\frac{p}{q}}=(a^{\frac{qm}{pn}+1})^{\frac{p}{q}}$.

9. $a^{\frac{m}{n}}\sqrt[p]{a^r}=\sqrt[p]{a^{\frac{pm}{n}+r}}$, $a^{m+1}\sqrt[m-1]{a}=\sqrt[m-1]{a^{(m+1)(m-1)+1}}$.

10. $b^{m-1}(b)^{\frac{1}{m-1}}=(b^{(m-1)^2+1})^{\frac{1}{m-1}}$, $a^{mn}(a^{-1})^{\frac{1}{mn}}=(a^{m^2n^2-1})^{\frac{1}{mn}}$.

49. To Transform the sum of two or more Radicals.

Radicals are said to be *Like* or *Similar* when they have the same indices and the same quantities under the radical signs; thus, $a\sqrt{b}$ and $\frac{1}{2}\sqrt{b}$ are like; $(a+b)\sqrt{a^mb}$ and $9a^2\sqrt{a^mb}$ are like; so also are $\sqrt[3]{\frac{a-b}{c}}$ and $5\sqrt[3]{\frac{a-b}{c}}$.

The algebraic sum of two or more like radical expressions may readily be transformed into a single term; thus,

$$3a\sqrt{bc}+a^2b\sqrt{bc}-2\sqrt{bc}=(3a-a^2b-2)\sqrt{bc}.$$

We have simply to take the algebraic sum of the quantities without the radicals, and after this write the common radical.

If the quantities without are numerals entirely, they are actually added or subtracted according to their signs; thus,

$$3\sqrt{a+b}+5\sqrt{a+b}=8\sqrt{a+b}.$$

$$7\sqrt{\frac{a}{b}}-5\sqrt{\frac{a}{b}}=2\sqrt{\frac{a}{b}}.$$

Radicals which are not like as given, may sometimes be made so; thus,

$$a\sqrt{b^3c}-b\sqrt{bcd^2}=ab\sqrt{bc}-bd\sqrt{bc}$$
$$=(ab-bd)\sqrt{bc}.$$

This operation of reducing radical expressions to a single term is generally called addition or subtraction of radicals, as the case may be.

Examples.

1. $3\sqrt{a}+2\sqrt{a}=5\sqrt{a}$, $\quad a\sqrt{a}+b\sqrt{a}=(a+b)\sqrt{a}$.

2. $8a^2x\sqrt{3c^2}+4a^2x\sqrt{3c^2}=12a^2x\sqrt{3c^2}$, $\quad 3a\sqrt{b}-5b\sqrt{b}=(3a-5b)\sqrt{b}$.

3. $5a(b)^{\frac{1}{2}}-2a(b)^{\frac{1}{2}}=3a(b)^{\frac{1}{2}}$, $\quad 3(a+b)^{\frac{1}{2}}-(a+b)^{\frac{1}{2}}=2(a+b)^{\frac{1}{2}}$.

4. $\frac{a}{b}\sqrt{\frac{c}{d}}-\frac{c}{d}\sqrt{\frac{c}{d}}=\left(\frac{a}{b}-\frac{c}{d}\right)\sqrt{\frac{c}{d}}$, $\quad \frac{a}{2+b}\sqrt{2a^2b}+\frac{1}{a}\sqrt{2a^2b}=\left(\frac{a}{2+b}+\frac{1}{a}\right)\sqrt{2a^2b}$.

5. $\frac{a}{b}\left(\frac{1}{c}\right)^{\frac{1}{m}}+\left(\frac{1}{c}\right)^{\frac{1}{m}}=\left(\frac{a}{b}+1\right)\left(\frac{1}{c}\right)^{\frac{1}{m}}$, $\quad \frac{3}{4}\left(-\frac{a}{b}\right)^{\frac{1}{3}}-\frac{1}{4}\left(-\frac{a}{b}\right)^{\frac{1}{3}}=\frac{1}{2}\left(-\frac{a}{b}\right)^{\frac{1}{3}}$.

6. $2a\sqrt{b^3c}-3b\sqrt{bc^3}=2ab\sqrt{bc}-3bc\sqrt{bc}=(2ab-3bc)\sqrt{bc}$.

7. $\frac{1}{2}(a^3b^3)^{\frac{1}{2}}-\frac{1}{3}(ab)^{\frac{1}{2}}=\frac{1}{2}ab(ab)^{\frac{1}{2}}-\frac{1}{3}(ab)^{\frac{1}{2}}=(\frac{1}{2}ab-\frac{1}{3})(ab)^{\frac{1}{2}}$.

8. $9\sqrt[3]{16a^{-1}}-10\sqrt[3]{54a^{-1}}=18\sqrt[3]{2a^{-1}}-30\sqrt[3]{2a^{-1}}=-12\sqrt[3]{2a^{-1}}$.

9. $5\left(\frac{16}{a}\right)^{\frac{1}{3}}-\left(\frac{54}{a}\right)^{\frac{1}{3}}=10\left(\frac{2}{a}\right)^{\frac{1}{3}}-3\left(\frac{2}{a}\right)^{\frac{1}{3}}=7\left(\frac{2}{a}\right)^{\frac{1}{3}}.$

10. $7\sqrt{\frac{a^2b}{4}}+5\sqrt{\frac{a^2b}{9}}=\frac{7a}{2}\sqrt{b}+\frac{5a}{3}\sqrt{b}=\frac{31a}{6}\sqrt{b}.$

11. $\sqrt{\frac{3}{4}}+\sqrt{\frac{3}{16}}=\frac{1}{2}\sqrt{3}+\frac{1}{4}\sqrt{3}=\frac{3}{4}\sqrt{3}.$

12. $\sqrt{\frac{a}{9}}+\sqrt{\frac{a}{36}}=\frac{1}{3}\sqrt{a}+\frac{1}{6}\sqrt{a}=\frac{1}{2}\sqrt{a}.$

50. General Principles of Exponents.

From the nature of exponents we have the following general results:

$$a^m \times a^n = a^{m+n}$$

$$\frac{a^m}{a^n}=a^{m-n}$$

$$(a^m)^n=a^{mn}$$

$$\sqrt[n]{a^m}=a^{\frac{m}{n}}.$$

Thus, we may say that,

I. *The addition of exponents takes place in the multiplication of quantities.*

II. *The subtraction of exponents takes place in the division of quantities.*

III. *The multiplication of exponents takes place in the formation of powers.*

IV. *The division of exponents takes place in the extraction of roots.*

SECTION V.

TRANSFORMATIONS OF POLYNOMIALS.

51. We have now pretty well disposed of the operations upon monomials or single terms; let us proceed to investigate the transformations which may be performed upon polynomials. If the principles already established are carefully borne in mind, we shall find little difficulty.

52. Multiplication of a Polynomial by a Monomial.

First, let a be the sum of all the positive terms in any polynomial, and b be the sum of all the negative terms in the same; the polynomial itself will be $a-b$.

The product of this polynomial by any single term, as c, will be

$$(a-b)c.$$

Let us now convert this product into an algebraic sum. The product of a and c is ac; but this is greater than the true value of the given expression, $(a-b)c$, since b should have been taken from a before it was multiplied by c. If, then, we subtract bc from ac, we shall have the true product; hence,

$$(a-b)c=ac-bc.$$

We should have obtained the same result by simply multiplying each term of the polynomial by c, observing the law of signs, and uniting the partial products by their respective signs.

If c had been negative we should have found the same principle to apply.

We may say, then, in general, that,

To multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial, remembering that like signs give plus, and unlike minus.

EXAMPLES.

1. $5a^2(3ab+2c-1)$. *Ans.* $15a^3b+10a^2c-5a^2$.

2. $\frac{1}{2}a^{\frac{1}{2}}(2a^{\frac{1}{2}}b^{-1}+\frac{1}{2})$. *Ans.* $ab^{-1}+\frac{1}{4}a^{\frac{1}{2}}$.

3. $3\sqrt{a}(5\sqrt{6}-\sqrt{a})$. *Ans.* $15\sqrt{6a}-3a$.

4. $\frac{a}{b}\left(\frac{a}{b}-\sqrt{a}+\frac{1}{4}\right)$. *Ans.* $\frac{a^2}{b^2}-\frac{a}{b}\sqrt{a}+\frac{a}{4b}$.

5. $\sqrt{c}(\sqrt[3]{c}-20+2\sqrt{c})$. *Ans.* $\sqrt[6]{c^5}-20\sqrt{c}+2c$.

6. $-25a^2x^4(1-3abx^{-1}-a^{\frac{1}{2}})$. *Ans.* $-25a^2x^4+75a^3bx^3+25a^{\frac{5}{2}}x^4$.

7. $2a^2\left(\frac{a+b}{c}-3a^{-2}\right)$. *Ans.* $\frac{2a^3+2a^2b}{c}-6$.

8. $\frac{\sqrt{a}}{\sqrt{b}}\left(\frac{\sqrt{a}}{\sqrt{b}}-\frac{\sqrt{c}}{\sqrt{d}}-\frac{\sqrt[3]{a^2}}{\sqrt[3]{b^2}}\right)$. *Ans.* $\frac{a}{b}-\frac{\sqrt{ac}}{\sqrt{bd}}-\frac{\sqrt[6]{a^7}}{\sqrt[6]{b^7}}$.

We are said to *factor* an expression when we transform it so that the several factors which enter it are made visible to the eye; thus, $ab+ac-ad=a(b+c-d)$. When we have an algebraic sum in which a monomial and a polynomial factor enters, the operation of *factoring* is the reverse of that in the above examples.

Let the student be required to factor the answers given in the above examples.

53. The Multiplication of Polynomials.

Resuming the polynomial, $a-b$, let $c-d$ be any other polynomial. Their product will be

$$(a-b)(c-d).$$

To transform this product into an algebraic sum, let us begin by multiplying $a-b$ by c; we shall have

$$ac-bc.$$

This result is d times greater than the product of the two polynomials, since d should have been subtracted from c before we used it as a multiplier. If we, then, subtract from $ac-bc$ the quantity $d(a-b)$, we shall have the true product of the polynomials; that is

$$(a-b)(c-d)=ac-bc-(ad-bd)=ac-bc-ad+bd.$$

It will be observed that we have simply multiplied each term of one polynomial by every term of the other. No mention was made of the signs in deducing this result, so that by inspection we see again, that like signs give plus, and unlike minus.

We may, then, say, that

To multiply one polynomial by another, multiply each term of the one by every term of the other, and simplify the result.

It is sometimes more convenient to set one polynomial under the other, and write like terms under each other as the operation proceeds; thus,

$$\begin{array}{lll}
a^2-ab+b^2 & & \\
a^2+ab+b^2 & & \\
\hline
a^4-a^3b+a^2b^2 & & \\
\quad a^3b-a^2b^2+ab^3 & & \\
\qquad a^2b^2-ab^3+b^4 & & \\
\hline
a^4 \quad +a^2b^2 \quad +b^4 & &
\end{array}$$

EXAMPLES.

1. $(3a-2b^2c-1)(5a^2c+b-x)$.
   Ans. $15a^3c-10a^2b^2c^2-5a^2c+3ab-2b^3c-b-3ax+2b^2cx+x$.

2. $\left(\frac{1}{2}a^{\frac{1}{2}}-b+\frac{a}{b}\right)\left(1-b-\frac{a}{b}\right)$.

   Ans. $\frac{1}{2}a^{\frac{1}{2}}-b+\frac{a}{b}-\frac{1}{2}a^{\frac{1}{2}}b+b^2-\frac{a^{\frac{3}{2}}}{2b}-\frac{a^2}{b^2}$.

3. $(\sqrt{a}+\frac{1}{2}\sqrt{b}-\frac{1}{2})(\sqrt{a}-\frac{1}{2}\sqrt{b}+\frac{1}{2})$. *Ans.* $a-\frac{1}{4}b+\frac{1}{2}\sqrt{b}-\frac{1}{4}$.

4. $(5a^2b^{-2}-1+c^{\frac{2}{3}})(a^2b+\frac{1}{4}-c^{\frac{1}{2}})$.

   Ans. $\frac{5a^4}{b}+\frac{5a^2}{4b^2}-\frac{5a^2c^{\frac{1}{2}}}{b^2}-a^2b-\frac{1}{4}+c^{\frac{1}{2}}+a^2bc^{\frac{2}{3}}+\frac{1}{4}c^{\frac{2}{3}}-c^{\frac{7}{6}}$.

5. $\left(\frac{a+b}{c}-1\right)\left(1-\frac{a+b}{c}\right)$. *Ans.* $-\frac{(a+b)^2}{c^2}+\frac{2(a+b)}{c}-1$.

6. $(a+b)(a-b)$. *Ans.* a^2-b^2.

7. $(a+b)(a+b)$. *Ans.* $a^2+2ab+b^2$.

8. $(a-b)(a-b)$. *Ans.* $a^2-2ab+b^2$.

9. $\left(\frac{a}{b}+\frac{c}{d}\right)\left(\frac{a}{b}-\frac{c}{d}\right)$. *Ans.* $\frac{a^2}{b^2}-\frac{c^2}{d^2}$.

10. $\left(\frac{\sqrt{a}}{\sqrt{b}}+\frac{\sqrt{c}}{\sqrt{d}}\right)\left(\frac{\sqrt{a}}{\sqrt{b}}+\frac{\sqrt{c}}{\sqrt{d}}\right)$. *Ans.* $\frac{a}{b}+\frac{2\sqrt{ac}}{\sqrt{bd}}+\frac{c}{d}$.

11. $\left(\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}-\frac{c^{\frac{1}{2}}}{d^{\frac{1}{2}}}\right)\left(\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}-\frac{c^{\frac{1}{2}}}{d^{\frac{1}{2}}}\right).$ *Ans.* $\frac{a}{b}-\frac{2a^{\frac{1}{2}}b^{\frac{1}{2}}}{b^{\frac{1}{2}}d^{\frac{1}{2}}}+\frac{c}{d}.$

12. $(\sqrt{a}+\sqrt[3]{ab}-\sqrt{b})(\sqrt{a}-\sqrt[3]{ab}+\sqrt{b}).$

Ans. $a-\sqrt[3]{a^2b^2}+2\sqrt[6]{a^2b^5}+b.$

13. $\left(\frac{a-b}{a+b}+\frac{a+b}{a-b}\right)\left(\frac{a-b}{a+b}+\frac{a+b}{a-b}\right).$ *Ans.* $\frac{(a-b)^2}{(a+b)^2}+\frac{(a+b)^2}{(a-b)^2}+2.$

14. $(-p+\sqrt{q+p^2})(-p-\sqrt{q+p^2}).$ *Ans.* $-q.$

54. Two Terms unchanged by Simplifying.

A polynomial is said to be *arranged* with respect to the descending powers of a particular letter, when the first term contains that letter with its highest exponent, and the next contains it with its next highest exponent, and so on. It is arranged according to the ascending powers of the letter when the order of terms is reversed.

Now, when two polynomials are multiplied together, there are always two terms in the result which cannot disappear in the process of simplification. These are, first, the term which results from combining the two terms with the highest exponents of a particular letter; and, second, the term which results from combining the two terms with lowest exponents of the same letter; thus,

$$(x^3-3x^2+x)(x^2-2x)=$$
$$x^5-5x^4+7x^3-2x^2.$$

In this case, x^5 and $-2x^2$ come immediately from the combination of certain terms; while the others are the results of simplifying.

55. The Division of Polynomials.

The dividend must always be produced by multiplying the quotient by the divisor; and when these are polynomials, we know, from the last article, that there must always be two terms at least in the dividend which will undergo no change from the process of simplification.

Let us take the polynomial $x^5-5x^4+7x^3-2x^2$, and let the divisor x^2-2x be given to find the quotient.

Now since x^5 must have resulted from the multiplication of x^2 in the divisor by that term in the quotient which contains the highest power of x, we can tell at once, by dividing x^5 by x^2, what that term must have been. One term of the quotient must then be x^3.

Let us for convenience arrange both polynomials, and write the dividend first and the divisor after it; thus,

$$\begin{array}{l|l}
x^5-5x^4+7x^3-2x^2 & x^2-2x \\
\underline{x^5-2x^4} & \overline{x^3-3x^2+x} \\
\quad -3x^4+7x^3-2x^2 & \\
\quad \underline{-3x^4+6x^3} & \\
\qquad\qquad x^3-2x^2 & \\
\qquad\qquad \underline{x^3-2x^2} & \\
\qquad\qquad\qquad 0 &
\end{array}$$

writing the first term of the quotient under the divisor. Now, as this term of the quotient must be multiplied into every term of the divisor, in the process of forming the dividend by multiplying the divisor and quotient together, if we perform this multiplication and subtract the result from the dividend, we shall free it from all partial products in which this term, x^2, of the quotient enters. Multiplying and subtracting, as shown above, we have a remainder, $-3x^4+7x^3-2x^2$.

Now, the term $-3x^4$ of the remainder must have resulted from multiplying x^2 by that term of the quotient which has the next highest power of x in it. We may thus find another term of the quotient by dividing the first term of the remainder by the first term of the divisor; thus we have, $-3x^2$. Writing this after the term of the quotient already found, and multiplying the divisor by it, as before, and subtracting the result from the remainder, we have a second remainder, x^3-2x^2. By continuing this process, we shall find all the terms of the quotient. If there should prove to be a final remainder, the division cannot be exactly performed. In this case, we may write the remainder in the form of a fraction, having the divisor for the denominator, and unite it with the quotient by its proper sign.

We may then say, practically, that

To divide one polynomial by another, arrange the dividend and divisor according to the powers of the same letter; divide the first term of the dividend by the first term of the divisor. This will be the first term of the quotient.

Multiply the divisor by this term of the quotient and subtract the result from the dividend.

Divide the first term of the remainder (arranged) by the first term of the divisor; multiply as before, and subtract from the remainder.

Continue the operation until there is no remainder, or until the first term of the remainder will not contain the first term of the divisor.

EXAMPLES.

1. $(a^2+2ab+b^2)\div(a+b)$. *Ans.* $a+b$.

2. $(a^2-2ab+b^2)\div(a-b)$. *Ans.* $a-b$.

3. $(a^2-b^2)\div(a+b)$. *Ans.* $a-b$.

4. $(4a^3+4a^2-29a+21)\div(2a-3)$. *Ans.* $2a^2+5a-7$.

5. $(a^4-b^4)\div(a-b)$. *Ans.* $a^3+a^2b+ab^2+b^3$.

6. $(x^5+y^5)\div(x+y)$. *Ans.* $x^4-x^3y+x^2y^2-xy^3+y^4$.

7. $(x^6+y^6)\div(x^2+y^2)$. *Ans.* $x^4-x^2y^2+y^4$.

8. $\dfrac{a^5-b^5}{a-b}$. *Ans.* $a^4+a^3b+a^2b^2+ab^3+b^4$.

9. $\dfrac{a^m-b^m}{a-b}$. *Ans.* $a^{m-1}+a^{m-2}b+a^{m-3}b^2+\cdots b^{m-1}$

10. $\dfrac{12y^4-192}{3y-6}$. *Ans.* $4y^3+8y^2+16y+32$.

11. $1\div(1-a)$. *Ans.* $1+a+a^2+a^3+a^4+\dfrac{a^5}{1-a}$.

12. $\dfrac{\sqrt[3]{a^2}+4\sqrt[6]{a^3}+8\sqrt[3]{a}+8\sqrt[6]{a}}{\sqrt[3]{a}+2\sqrt[6]{a}}$. *Ans.* $\sqrt[3]{a}+2\sqrt[6]{a}+4$.

13. $\dfrac{a+a^{\frac{1}{2}}b^{\frac{1}{2}}+b}{a^{\frac{1}{2}}+a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}}}$. *Ans.* $a^{\frac{1}{2}}-a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}}$.

14. $\left(\dfrac{a}{b}-\dfrac{c}{d}\right)\div\left(\dfrac{\sqrt{a}}{\sqrt{b}}-\dfrac{\sqrt{c}}{\sqrt{d}}\right)$. *Ans.* $\dfrac{\sqrt{a}}{\sqrt{b}}+\dfrac{\sqrt{c}}{\sqrt{d}}$.

15. $\dfrac{a^{\frac{1}{2}}+2a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}}}{a^{\frac{1}{4}}+b^{\frac{1}{4}}}$. *Ans.* $a^{\frac{1}{4}}+b^{\frac{1}{4}}$.

16. $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt[4]{a}-\sqrt[4]{b}}$. *Ans.* $\sqrt[4]{a}+\sqrt[4]{b}$.

56. Formulas used in Transformations.

A *Formula* is any general truth expressed by means of symbols.

The transformation of polynomials can often be much shortened by the use of certain simple formulas. We shall now give a few of these.

$$(a+b)^2=(a+b)(a+b)=a^2+2ab+b^2\text{ ; that is,}$$

The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first and second, plus the square of the second ; and the converse.

$$(a-b)^2=(a-b)(a-b)=a^2-2ab+b^2\text{ ; that is,}$$

The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first and second, plus the square of the second ; and the converse.

$$(a+b)(a-b)=a^2-b^2, \therefore$$

The product of the sum and difference of two quantities is equal to the difference of their squares, and the converse.

When the second member of an equation is the algebraic sum resulting from operations indicated in the first, such sum is called the *development* of the first member; thus, in

$$(a+b)^2=a^2+2ab+b^2,$$

the second member is the development of the first.

Examples.

Develop the following expressions, by the aid of the foregoing formulas.

1. $(x+y)^2$. *Ans.* $x^2+2xy+y^2$.
2. $(2a+2b)^2$. *Ans.* $4a^2+8ab+4b^2$.
3. $(a-1)^2$. *Ans.* a^2-2a+1.
4. $(\frac{1}{2}-a)^2$. *Ans.* $\frac{1}{4}-a+a^2$.
5. $(\frac{1}{3}a+\frac{1}{2}b)^2$. *Ans.* $\frac{1}{9}a+\frac{1}{3}ab+\frac{1}{4}b^2$.
6. $\left(\frac{\sqrt{a}}{\sqrt{b}}-1\right)^2$. *Ans.* $\frac{a}{b}-\frac{2\sqrt{a}}{\sqrt{b}}+1$.

7. $\left(\dfrac{(a+b)^{\frac{1}{2}}}{(a-b)^{\frac{1}{2}}}-a^{\frac{1}{2}}\right)^2.$ *Ans.* $\dfrac{a+b}{a-b}-\dfrac{2a^{\frac{1}{2}}(a+b)^{\frac{1}{2}}}{(a-b)^{\frac{1}{2}}}+a.$

8. $(p-\sqrt{q+p^2})^2.$ *Ans.* $p^2-2p\sqrt{q+p^2}+(q+p^2).$

9. $(1-a^{-1})^2.$ *Ans.* $1-2a^{-1}+a^{-2}.$

10. $(x+y)(x-y).$ *Ans.* $x^2-y^2.$

11. $(\frac{1}{2}a-b)(\frac{1}{2}a+b).$ *Ans.* $\frac{1}{4}a^2-b^2.$

12. $(3a-\frac{1}{2}b)(3a+\frac{1}{2}b).$ *Ans.* $9a^2-\frac{1}{4}b^2.$

13. $\left(\dfrac{a}{b}+\dfrac{c}{d}\right)\left(\dfrac{a}{b}-\dfrac{c}{d}\right).$ *Ans.* $\dfrac{a^2}{b^2}-\dfrac{c^2}{d^2}.$

14. $(\sqrt{a}+b)(\sqrt{a}-b).$ *Ans.* $a-b^2.$

15. $(1-a)(1+a).$ *Ans.* $1-a^2.$

16. $(1-\frac{1}{2}a)(1+\frac{1}{2}a).$ *Ans.* $1-\frac{1}{4}a^2.$

17. $(a^{2m}+b^{4m})(a^{2m}-b^{4m}).$ *Ans.* $a^{4m}-b^{8m}.$

18. $(\sqrt[m]{a}-\sqrt[n]{b})(\sqrt[m]{a}+\sqrt[n]{b}).$ *Ans.* $\sqrt[m]{a^2}-\sqrt[n]{b^2}.$

57. Formulas used in Transformations—*Continued.*

By developing the following, we have

$$(a+b)(a+c)=a^2+(b+c)a+bc$$
$$(a-b)(a-c)=a^2-(b+c)a+bc.$$

Whence we see that when two binomials are to be multiplied together, having their first terms the same and the second terms different, with the same middle signs, we may write out the result at once:

1. The first term of the result must be the square of the common term;

2. The second term must be the numerical sum of the two last terms into the first term with the sign of the last terms; and

3. The last term must be the product of the last terms with the plus sign.

Examples.

1. $(a+2)(a+3)=a^2+5a+6.$

2. $(x-5)(x-4)=x^2-9x+20.$

3. $(1+a)(1+b)=1+(a+b)+ab.$

4. $(a^2b+5)(a^2b+7)=a^4b^2+12a^2b+35.$

5. $(3a-\frac{1}{2})(3a-\frac{1}{4})=9a^2-\frac{9}{4}a+\frac{1}{8}.$

6. $\left(\frac{a}{b}+2\right)\left(\frac{a}{b}+9\right)=\frac{a^2}{b^2}+\frac{11a}{b}+18.$

7. $\left(\frac{\sqrt{a}}{\sqrt{b}}-1\right)\left(\frac{\sqrt{a}}{\sqrt{b}}-20\right)=\frac{a}{b}-\frac{21\sqrt{a}}{\sqrt{b}}+20.$

8. $(a^{-2}-1)(a^{-2}-5)=a^{-4}-6a^{-2}+5.$

58. Formulas used in Transformations—*Continued.*

Now let the last terms be different with different signs; thus,

$$(a+b)(a-c)=a^2+(b-c)a-bc.$$

Whence we see that in this case the middle term will be the difference of the last terms into the common term, and the last term will have the $-$ sign.

Examples.

1. $(a+3)(a-5)=a^2-2a-15.$

2. $(a^{\frac{1}{2}}-9)(a^{\frac{1}{2}}+3)=a-6a^{\frac{1}{2}}-27.$

3. $(3a^2b^4-2)(3a^2b^4+8)=9a^4b^8-18a^2b^4-16.$

4. $(\sqrt{a}-1)(\sqrt{a}+\frac{1}{2})=a-\frac{1}{2}\sqrt{a}-\frac{1}{2}.$

5. $(\sqrt[3]{x}-5)(\sqrt[3]{x}+7)=\sqrt[3]{x^2}+2\sqrt[3]{x}-35.$

6. $(a^{-1}+9)(a^{-1}-11)=a^{-2}-2a^{-1}-99.$

7. $(a^m-4)(a^m+12)=a^{2m}+8a^m-48.$

8. $(x^{\frac{m}{n}}-9)(x^{\frac{m}{n}}+2)=x^{\frac{2m}{n}}-7x^{\frac{m}{n}}-18.$

9. $(a^{\frac{3}{4}}-1)(a^{\frac{3}{4}}+25)=a^{\frac{3}{2}}+24a^{\frac{3}{4}}-25.$

59. Factoring.

By the aid of the simple formulas already explained, we may often resolve trinomials into their factors by mere inspection. Let us first take some examples under the three following formulas:

$$a^2+2ab+b^2=(a+b)(a+b).$$

$$a^2-2ab+b^2=(a-b)(a-b).$$

$$a^2-b^2=(a+b)(a-b).$$

EXAMPLES.

1. $a^2b^2-c^2=(ab+c)(ab-c).$
2. $4a^2-4a+1=(2a-1)(2a-1).$
3. $x^2+x+\frac{1}{4}=(x+\frac{1}{2})(x+\frac{1}{2}).$
4. $x^2-\frac{x}{2}+\frac{1}{16}=(x-\frac{1}{4})(x-\frac{1}{4}).$
5. $a-b=(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}.$
6. $4a^2-\frac{1}{4}=(2a-\frac{1}{2})(2a+\frac{1}{2}).$
7. $25a^2b^4+20ab^2c+4c^2=(5ab^2+2c)(5ab^2+2c).$
8. $\frac{25a^{-2}}{4b^2}-\frac{x}{y}=\left(\frac{5a^{-1}}{2b}+\frac{\sqrt{x}}{\sqrt{y}}\right)\left(\frac{5a^{-1}}{2b}-\frac{\sqrt{x}}{\sqrt{y}}\right).$
9. $a^{\frac{1}{2}}-b^{\frac{1}{4}}=(a^{\frac{1}{4}}+b^{\frac{1}{8}})(a^{\frac{1}{4}}-b^{\frac{1}{8}}).$
10. $\sqrt{x}-\frac{1}{4}=(\sqrt[4]{x}+\frac{1}{2})(\sqrt[4]{x}-\frac{1}{2}).$

60. Factoring—*Continued.*

Let us now show the operation of factoring under the formulas,

$$(a+b)(a+c)=a^2+(b+c)a+bc,$$

$$(a-b)(a-c)=a^2-(b+c)a+bc:$$

For example, take

$$a^2+5a+6.$$

If there are any binomial factors in this expression, the product of their first terms must produce a^2, and, since a enters the middle term, they must both be a : since the last term is positive the last terms must have like signs. Looking at the middle term we see that they must be + ; we may then write so much of the required factors ; thus,

$$a+\quad,\ a+\quad.$$

Now we must have two such quantities for the second terms, that when multiplied together they will produce 6, the last term of the trinomial, and when added will give 5. These terms, then, can only be +2 and +3, and hence the factors are $a+2$ and $a+3$, and we have

$$a^2+5a+6=(a+2)(a+3).$$

Again, take

$$a^2-5a+6.$$

Here the middle term being $-$, the last terms of the factors must be $-$ also; thus,

$$a^2-5a+6=(a-2)(a-3).$$

When the last sign in the trinomial is $-$, thus;

$$a^2+2a-8,$$

the last terms must be unlike to give a $-$ quantity, and the $+$ must be the greater to give $+2$ in the middle term. Those terms must thus be, $+4$ and -2, and we have,

$$a^2+2a-8=(a+4)(a-2).$$

Examples.

1. $x^2+3x+2=(x+2)(x+1)$.
2. $x^2+17x+72=(x+8)(x+9)$.
3. $x^2-12x+35=(x-7)(x-5)$.
4. $x^2+24x-25=(x+25)(x-1)$.
5. $x^2-\frac{3}{4}x+\frac{1}{8}=(x-\frac{1}{2})(x-\frac{1}{4})$.
6. $a^2x^2-74ax-75=(ax+1)(ax-75)$.
7. $9a^2+12a-5=(3a-1)(3a+5)$.
8. $\frac{a^2}{b^2}-10\frac{a}{b}+25=\left(\frac{a}{b}-5\right)\left(\frac{a}{b}-5\right)$.
9. $\frac{a}{b}+\frac{3a^{\frac{1}{2}}}{b^{\frac{1}{2}}}-54=\left(\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}+9\right)\left(\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}-6\right)$.
10. $\sqrt{a}-3\sqrt[4]{a}+2=(\sqrt[4]{a}-1)(\sqrt[4]{a}-2)$.
11. $a^{-2m}+5a^{-m}+6=(a^{-m}+3)(a^{-m}+2)$.
12. $25x^2-60x+35=(5x-7)(5x-5)$.

Remark.—When there is a monomial factor present, remove it first and then factor.

13. $a^2x^2-12a^2x+35a^2=a^2(x-7)(x-5)$.
14. $5a^2+15a+10=5(a+1)(a+2)$.
15. $2x^2y^2-\frac{6xy^2}{4}+\frac{2y^2}{8}=2y^2(x-\frac{1}{2})(x-\frac{1}{4})$.
16. $a^2\sqrt{b}-3a\sqrt{b}+2\sqrt{b}=\sqrt{b}(a-1)(a-2)$.

61. The Division of the Difference of like Powers.

The difference of the like powers of any two quantities is always divisible by the difference of the quantities themselves.

For, let a and b be any two quantities, and m be any positive whole number. The difference of the like powers will be $a^m - b^m$.

Beginning the division, we have,

$$\begin{array}{l|l} a^m - b^m & a-b \\ \cline{2-2} a^m - a^{m-1}b & a^{m-1} \\ \hline \quad a^{m-1}b - b^m, \text{ or} & \\ \quad b(a^{m-1} - b^{m-1}). & \end{array}$$

Now, if this remainder is exactly divisible by $a-b$, then, $a^m - b^m$ is itself exactly divisible by it. But the remainder is b times $a^{m-1} - b^{m-1}$, so that if the factor $a^{m-1} - b^{m-1}$ is exactly divisible by $a-b$, the entire remainder is divisible by this difference, and hence the dividend $a^m - b^m$ is likewise so divisible. We see, thus, that if $a^{m-1} - b^{m-1}$ is divisible by $a-b$, $a^m - b^m$ is also divisible by the same quantity; that is to say,

If the difference of the like powers of two quantities is exactly divisible by the difference of the quantities themselves, then, the difference of such powers greater by unity, is also exactly divisible by the difference of the quantities.

But we know that $a^2 - b^2$ is exactly divisible by $a-b$; hence, from the hypothetical proposition just established, $a^3 - b^3$ must likewise be so divisible, and hence $a^4 - b^4$ must be, and so on to infinity; which was to be proved.

This method of proof is called *Mathematical Induction.*

The form of the quotient will be,

$$\frac{a^m - b^m}{a-b} = a^{m-1} + a^{m-2}b + a^{m-3}b^2 + \cdot \cdot \cdot \cdot \cdot ab^{m-2} + b^{m-1}.$$

62. The Division of like Powers.

The following propositions may also be readily demonstrated:

1. *The difference of the like* EVEN *powers of two quantities, is always divisible by the sum of the quantities.* The form of the quotient will be,

$$\frac{a^{2m} - b^{2m}}{a+b} = a^{2m-1} - a^{2m-2}b + a^{2m-3}b + \cdot \cdot \cdot \cdot - b^{2m-1}.$$

$$\frac{a^4 - b^4}{a+b} = a^3 - a^2b + ab^2 - b^3.$$

2. *The sum of the like* ODD *powers of two quantities is always divisible by the sum of the quantities themselves.* The form of the quotient will be,

$$\frac{a^{2m+1}+b^{2m+1}}{a+b}=a^{2m}-a^{2m-1}b+a^{2m-2}b^2-\cdots+b^{2m}.$$

$$\frac{a^5+b^5}{a+b}=a^4-a^3b+a^2b^2-ab^3+b^4.$$

Examples.

Write out the developments of the quotients indicated.

1. $\dfrac{x^2-y^2}{x-y}=x+y,\quad \dfrac{x^3-y^3}{x-y}=x^2+xy+y^2.$

2. $\dfrac{x^3+y^3}{x+y}=x^2-xy+y^2,\quad \dfrac{x^4-y^4}{x+y}=x^3-x^2y+xy^2-y^3.$

3. $\dfrac{8a^3b^3-c^3}{2ab-c}=4a^2b^2+2abc+c^2.$

4. $\dfrac{a^5+32}{a+2}=a^4-2a^3+4a^2-8a+16.$

5. Factor a^6-b^6, $x^{\frac{2}{3}}-1$, $1-x^3$, $8a^3-27$.

6. a^5b-9b^3, $12x^4-192$, $\dfrac{a^4-b^4}{c}$.

7. $a^{-3}-b^{-3}$, $\dfrac{1}{a^4-b^4}=\dfrac{1}{(a^2-b^2)(a^2+b^2)}=\dfrac{1}{(a-b)(a+b)(a^2+b^2)}$.

8. $\dfrac{a^8-b^8}{a^2+2ab+b^2}=\dfrac{(a^4-b^4)(a^4+b^4)}{(a+b)(a+b)}=\dfrac{(a^2+b^2)(a^2-b^2)(a^4+b^4)}{(a+b)(a+b)}=$
$\dfrac{(a^2+b^2)(a+b)(a-b(a^4+b^4)}{(a+b)(a+b)}$.

63. The Binomial Formula.

The process of raising a binomial to any power may be greatly shortened by using what is called the *Binomial Formula.* By actual multiplication we have the following developments:

1st Power		$(a+b)=a+b$
2d	"	$(a+b)^2=a^2+2ab+b^2$
3d	"	$(a+b)^3=a^3+3a^2b+3ab^2+b^3$

4th Power $(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$

5th " $(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$.

Examining any one of these developments, as the last, we see that a (called the leading letter) appears in the first term with an exponent equal to the exponent of the power of the binomial in that case, and that this exponent goes on diminishing by unity to the last term, in which it enters to the zero power, that is, not at all; thus,

$$a^5 \quad a^4 \quad a^3 \quad a^2 \quad a^1 \quad a^\circ.$$

With b the order is just reversed; thus,

$$b^\circ \quad b^1 \quad b^2 \quad b^3 \quad b^4 \quad b^5.$$

It will be observed, further, that any numerical co-efficient may be found from the preceding term by multiplying the co-efficient of that term by the exponent of a, the leading letter in that term, and dividing this product by the number of terms preceding the required term; thus, to find the co-efficient of the fourth term, in the development of the fifth power above, multiply 10, the co-efficient of the third term, by 3, the exponent of a in that term, and divide the product 30 by 3, the number of terms preceding the fourth, and we have 10, the co-efficient required. In like manner, the co-efficient of any other term may be found, remembering that the co-efficient of the first term is always unity. Generalizing, we may write,

$$(a+b)^m=a^m+ma^{m-1}b+\frac{m(m-1)}{1\cdot2}a^{m-2}b^2+$$

$$\frac{m(m-1)(m-2)}{1\cdot2\cdot3}a^{m-3}b^3+\cdots\cdots+b^m.$$

This is *The Binomial Formula*, and it may be rigidly demonstrated to be true, whether m be *entire* or *fractional*, *positive* or *negative*. When m is fractional or negative, the number of terms will be infinite.

The power of any binomial may be developed by means of this formula. For example, let us take,

$$(3x^2-2ab^2)^4.$$

Let $3x^2=a$ and $-2ab^2=b$.

We have at once,

$$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4.$$

Now in this, writing for a and b their values as above, we have,
$(3x^2-2ab^2)^4=(3x^2)^4+4(3x^2)^3(-2ab^2)+6(3x^2)^2(-2ab^2)^2+4(3x^2)(-2ab^2)^3+(-2ab^2)^4$.

$(3x^2-2ab^2)^4=81x^8-216x^6ab^2+216x^4a^2b^4-96x^2a^3b^6+16a^4b^8$.

EXAMPLES.

1. Develop $(a^2+1)^3$, $(ab^2-2)^4$, $\left(\frac{a}{b}+\frac{c}{d}\right)^3$.

2. Develop $(y-1)^4$, $(\sqrt{a}-1)^3$, $(\sqrt{a}+\sqrt{b})^4$.

3. Develop $(a+b)^{\frac{1}{2}}$, $(a+b)^{\frac{1}{2}}=a^{\frac{1}{2}}+\frac{1}{2}a^{\frac{1}{2}-1}b-\frac{1}{8}a^{\frac{1}{2}-2}b^2+\text{etc.}$

4. Develop $(a+b)^{-1}$, $(a+b)^{-1}=a^{-1}-a^{-2}b+a^{-3}b^2-\text{etc.}$

5. Develop $(a^{\frac{1}{2}}-1)^3$, $\left(x^{-1}+y^{\frac{1}{2}}\right)^3$, $(x-y)^{\frac{3}{4}}$.

6. Develop $(2-3x)^{-2}$, $\left(\frac{1}{a}-\frac{1}{b}\right)^3$, $(\sqrt{a}-\sqrt{b})^3$.

7. Develop $(1-a)^{\frac{1}{2}}$, $(a+1)^{-3}$, $(a+b)^{\frac{1}{n}}$.

64. The Powers of Polynomials.

The power of any polynomial may be developed by the use of the binomial formula. For example, let it be required to find the third power of $2a^2-4ab+3c^2$.

Let $2a^2-4ab=a$ and $3c^2=b$; then

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3.$$

Replacing the values of a and b, we have
$(2a^2-4ab+3c^2)^3=(2a^2-4ab)^3+3(2a^2-4ab)^2(3c^2)+3(2a^2-4ab)(3c^2)^2+(3c^2)^3$.

EXAMPLES.

1. Develop $(a+b+c)^3$.
2. Develop $(1-2x-3x^2)^2$.
3. Develop $(a^m+b^n)^3$.
4. Develop $(a^2+1+a^{-2})^2$.
5. Develop $(\sqrt{a}+1-\sqrt{b})^3$.

65. Common Multiple.

The *Common Multiple*, or a *Common Dividend* of two or more quantities, is any quantity which is exactly divisible by each of them: such a dividend may always be found by multiplying the several quantities together; thus, $4a^2$, $3ab$ and $3b^2c$ would give $36a^3b^3c$ for a Common Dividend.

The *Least Common Multiple* is the least quantity which is so divisible; thus, $12a^2b^2c$, is the least Common Multiple, or Dividend, of the quantities above given.

A common multiple must obviously contain every factor which enters any of the quantities by which it is to be divisible, as many times as such factor enters any one of the quantities; hence, if all the factors are made to appear, we have but to take each factor the greatest number of times it enters any one of the quantities, and multiply the results together; thus, $a^2+2ab+b^2$, a^2-b^2 and $a^2-2ab+b^2$, resolved into their simplest factors give,

$$(a+b)(a+b),\ (a+b)(a-b),\ (a-b)(a-b).$$

Here $(a+b)$ enters the first expression twice, and $(a-b)$ enters the third twice: the second expression containing no other factors than these, we have,

$$(a+b)^2(a-b)^2=2a^2+2b^2.$$

It is commonly better, however, to retain the indicated product as in the first member of this equation, than to develop it as in the second member. Thus, we may say that,

To find the Least Common Multiple of two or more quantities, resolve the quantities into their simple factors, and take each factor the greatest number of times it is found in any one of the expressions. The product of these factors will be the multiple required.

Examples.

Find the Least Common Multiple of the following:

1. a^2-b^2, $ab+b^2$, a^2-ab, $(a+b)^2$, $(a+b)(a-b)$, $b(a+b)$, $a(a-b)$, $(a+b)^2$. *Ans.* $ab(a+b)^2(a-b)$.

2. x^2+x-2, x^2+2x-3. *Ans.* $(x-1)(x+2)(x+3)$.

3. $(x^2-a)^2$, x^2-a, 5. *Ans.* $5(x^2-a)^2$.

4. $x^2+9x+20$, x^2+3x-4, x^2+4x-5.
Ans. $(x+4)(x+5)(x-1)$.

5. $6a^2-23a+7,\ 6a^2-25a+14.$ *Ans.* $(2a-7)(3a-1)(3a-2)$.

6. $3a,\ (a+b)^2,\ a^2-2ab+b^2.$ *Ans.* $3a(a+b)^2(a-b)^2$.

66. Operations upon Algebraic Fractions.

The student is supposed to be already familiar with the management of fractions in arithmetic. Their treatment in algebra is altogether the same. The transformation, however, of the sum or difference of two fractions into a single expression, requires some care, and we shall now give a number of examples for practice.

Let the student remember to first convert the fractions into equivalent fractions with a common fractional unit, and then to add or subtract the numerators, placing the result over the common denominator.

EXAMPLES.

1. $$\frac{1}{a+b}+\frac{1}{a-b}=\frac{a-b}{a^2-b^2}+\frac{a+b}{a^2-b^2}=\frac{a-b+a+b}{a^2-b^2}=\frac{2a}{a^2-b^2}.$$

2. $$\frac{1}{x+1}-\frac{2}{x+2}=\frac{x+2}{(x+1)(x+2)}-\frac{2(x+1)}{(x+1)(x+2)}=\frac{x+2-(2(x+1))}{(x+1)(x+2)}=$$
$$\frac{-x}{(x+1)(x+2)}.$$

3. $$\frac{1-a}{a^2-b^2}-\frac{1+a}{(a+b)^2}=\frac{(1-a)(a+b)-(1+a)(a-b)}{(a-b)(a+b)^2}=$$
$$\frac{a+b-a^2-ab-a+b-a^2+ab}{(a-b)(a+b)^2}=\frac{2b-2a^2}{(a-b)(a+b)^2}.$$

4. $$\frac{1-a}{1+a}-\frac{1+a}{1-a}+\frac{1}{(1+a)^2}=\frac{(1-a)(1-a^2)-(1+a)(1+a)^2+1-a}{(1+a)(1+a)(1-a)}=$$
$$\frac{1-5a-4a^2}{(1-a^2)(1+a)}.$$

5. $\dfrac{a}{a+b}+\dfrac{2b}{a-b}+\dfrac{b}{a+b}.$ *Ans.* $\dfrac{a+b}{a-b}.$

6. $\dfrac{a+3}{5}+\dfrac{2a-5}{3a}.$ *Ans.* $\dfrac{3a^2+19a-25}{15a}.$

7. $\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}.$ *Ans.* $\dfrac{4ab}{a^2-b^2}.$

8. $3x+\frac{x}{b}-\left(x-\frac{x-a}{c}\right).$ *Ans.* $2x+\frac{cx+bx-ab}{bc}.$

9. $\frac{1}{a^m+b^m}+\frac{a^m-b^m}{a^{2m}-b^{2m}}.$ *Ans.* $\frac{2a^m-2b^m}{a^{2m}-b^{2m}}.$

10. $\frac{1}{a^{\frac{1}{2}}-b^{\frac{1}{2}}}-\frac{1}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}.$ *Ans.* $\frac{2b^{\frac{1}{2}}}{a-b}.$

11. $\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}-\frac{\sqrt{b}}{\sqrt{a}+\sqrt{b}}.$ *Ans.* $\frac{a+b}{a-b}.$

12. $\frac{a^{-1}}{a-b}+\frac{b^{-1}}{a+b}.$ *Ans.* $\frac{a^2+b^2}{ab(a^2-b^2)}.$

67. Essential Sign.

The sign by which one expression is connected with another is called the *Sign of Operation;* the sign resulting from the combination of this sign with the sign of the quantity itself is called the *Essential Sign* of the expression: thus, in the expression

$$a-(-b),$$

the second term is to be subtracted from the first, and the sign of operation is $-$; but removing the parenthesis the two $-$ signs combine and give $+b$. This resulting sign is the essential sign of the term.

We sometimes cannot tell the essential sign of an expression absolutely; thus, in

$$x=\frac{a}{b-c},$$

the essential sign of the fraction, and so the sign of x, will depend upon the relative values of b and c. If $b>c$ it is $+$; if $b<c$ it is $-$.

EXAMPLES.

1. What is the essential sign of x in $x=-\frac{a}{b-c}$, when a is negative and $b>c$? when $b<c$?

2. What is the essential sign of x in $x=\frac{a-b}{c-d}$, when $a>b$ and $c<d$? when $a<b$ and $c<d$? when $a<b$ and $c>d$? when $a>b$ and $c>d$?

68. Imaginary Quantities.

We have seen that *the even root of a negative quantity is imaginary.* Let us now take an imaginary quantity of the second degree, as $\sqrt{-a}$, and multiply it by itself, applying the general law of signs; we shall have,

$$\sqrt{-a} \times \sqrt{-a} = \sqrt{a^2} = \pm a.$$

But the square of the square root of a quantity is, from the definition, the quantity itself: so that,

$$(\sqrt{-a})^2 = -a.$$

We should thus have $-a = \pm a$, or, taking the upper sign, $-a = +a$, which is impossible.

To avoid this difficulty we must modify the law of signs in the multiplication of imaginary expressions.

Now, every imaginary expression of the second degree may be put under the form of,

$$a\sqrt{-1}:$$

in which a represents any *real* quantity, whether its exact value may be found or not; and $\sqrt{-1}$, an imaginary quantity, which is called *the imaginary factor.*

For example, let us have,

$$\sqrt{-ab}.$$

This may be written,

$$\sqrt{-ab} = \sqrt{ab \times (-1)} = \sqrt{ab} \,.\, \sqrt{-1}.$$

Again,

$$\sqrt{-(a+b)^2} = \sqrt{(a+b)^2}.\sqrt{-1} = (a+b)\sqrt{-1}.$$

Let us now multiply $\sqrt{-1}$ by itself any number of times, until we discover the law of such combinations.

Since the square of the square root is the quantity itself, $\sqrt{-1} \times \sqrt{-1} = (\sqrt{-1})^2 = -1.$

The third power will be found by multiplying the second by the first; hence,

$$(\sqrt{-1})^2 \times \sqrt{-1} = (-1)\sqrt{-1} = -\sqrt{-1}.$$

In like manner,

$$(\sqrt{-1})^4 = (\sqrt{-1})^2(\sqrt{-1})^2 = (-1)(-1) = +1.$$

We may in the same way continue the operation, and, for the several powers of $\sqrt{-1}$, we shall have,

$$(\sqrt{-1})^1 = \qquad \sqrt{-1}.$$
$$(\sqrt{-1})^2 = \qquad -1.$$
$$(\sqrt{-1})^3 = (\sqrt{-1})^2(\sqrt{-1}) = -1\sqrt{-1} = \qquad -\sqrt{-1}.$$
$$(\sqrt{-1})^4 = (\sqrt{-1})^2(\sqrt{-1})^2 = (-1)(-1) = \qquad +1.$$
$$(\sqrt{-1})^5 = (\sqrt{-1})^2(\sqrt{-1})^3 = (-1)(-\sqrt{-1}) = \sqrt{-1}.$$
$$(\sqrt{-1})^6 = (\sqrt{-1})^2(\sqrt{-1})^4 = (-1)(+1) = \qquad -1.$$
$$(\sqrt{-1})^7 = (\sqrt{-1})^6(\sqrt{-1}) = (-1)(\sqrt{-1}) = \quad -\sqrt{-1}.$$
$$(\sqrt{-1})^8 = (\sqrt{-1})^4(\sqrt{-1})^4 = (+1)(+1) = \qquad +1.$$

It will be observed that the last four results are but a repetition of the preceding four, and so they would continue to repeat themselves in sets of fours.

The multiplication of imaginary quantities is effected by the use of this *imaginary factor;* that is, we first resolve each expression into two factors, one real and the other the imaginary factor, $\sqrt{-1}$; we then combine the real factors by the ordinary laws, and the imaginary factors according to the laws just deduced for the formation of its several powers.

If, however, only one of the quantities is imaginary, the ordinary rules apply.

Examples.

1. Multiply $5\sqrt{-4}$ by $2\sqrt{-4}$.

$$5\sqrt{4}\sqrt{-1} \times 2\sqrt{4}\sqrt{-1}$$
$$10\sqrt{-1} \times 4\sqrt{-1}$$
$$40(\sqrt{-1})^2 = -40. \text{ } Ans.$$

2. Multiply $a\sqrt{-b}$ by $c\sqrt{-d}$.

$$a\sqrt{b}\cdot\sqrt{-1} \times c\sqrt{d}\cdot\sqrt{-1}$$
$$ac\sqrt{bd}\cdot(\sqrt{-1})^2$$
$$-ac\sqrt{bd}. \text{ } Ans.$$

3. Multiply $(a+b)\sqrt{-(a+b)}$ by $(a+b)\sqrt{-(a+b)}$.

Ans. $-(a+b)^3$.

4. $\sqrt{-a}\cdot\sqrt{-b}\cdot\sqrt{-c} = -\sqrt{abc}\cdot\sqrt{-1}$.

5. $\sqrt{-2}\cdot\sqrt{-2}\cdot\sqrt{-2}=-2\sqrt{2}\sqrt{-1}$.

6. $(a-\sqrt{-b})(a+\sqrt{-b})=a^2+b$.

7. $(a+\sqrt{-b})(a+\sqrt{-b})=a^2+2a\sqrt{-b}-b$.

8. $3\sqrt{5}\times 5\sqrt{-2}=15\sqrt{-10}$.

9. $(3+\sqrt{-5})(2-\sqrt{-5})=11-\sqrt{-5}$.

10. $\sqrt{-a^2}\cdot\sqrt{-b^2}\cdot\sqrt{-c^2}\cdot\sqrt{d^2}\cdot\sqrt{-f^2}=abcdf$.

11. $(\sqrt{-3}+\sqrt{-1})(\sqrt{-12}-\sqrt{-4})=-4$.

12. $(\sqrt{-2}-\sqrt{3})(\sqrt{2}-\sqrt{-3})=\sqrt{-9}+\sqrt{-4}$.

The division of imaginary quantities of the second degree is managed in a corresponding way, being careful to introduce the imaginary factor $\sqrt{-1}$.

Examples.

1. Divide $10\sqrt{-1}$ by $5\sqrt{-1}$. *Ans.* 2.

2. Divide $a\sqrt{-b}$, by $c\sqrt{-d}$. *Ans.* $\dfrac{a\sqrt{b}}{c\sqrt{d}}$.

3. Divide $4\sqrt{-a^2b^2}$, by $2\sqrt{-b^2}$. *Ans.* $2a$.

4. Divide $9\sqrt{-10}$ by $3\sqrt{-2}$. *Ans.* $3\sqrt{5}$.

5. Divide a^2+b by $a-\sqrt{-b}$. *Ans.* $a+\sqrt{-b}$.

Imaginary expressions of a higher degree than the second, may be treated in a corresponding manner.

SECTION VI.

THE ROOTS OF NUMBERS.

69. The extraction of the roots of numbers properly belongs to Arithmetic; but it may be well to show here the rationale of the process.

Since the square root of 100 is 10, we know that the square root of any number which contains but two digits, must be expressed by a single figure. We can readily find the root of such a number by trial, when it is a perfect power.

When the number contains three or more figures, there will be at least two figures in its root; that is, the root will contain a certain number of *tens* and a certain number of *units*.

Then, let n be any number, and a the number of tens, and b the number of units in its root. The square root of n will be $(a+b)$, and we may write,

$$n=(a+b)^2=a^2+2ab+b^2 \quad \cdots\cdots \quad (1).$$

We see from this that the number contains a^2, that is, *the square of the tens*, and $2ab$, that is, *twice the product of the tens and units;* and b^2, *the square of the units.*

Let it now be required to find the square root of 1444. We shall begin by finding the number of tens in the root; and since the square of a single ten gives units followed by two 0's; thus, $(10)^2=100$, the square of any number of tens can give only 0's in the last two places. We may then place a point over the third figure from the right, as here seen, to show that the last two places, 44, may be regarded as occupied by 0's, in our search for the tens of the root.

$$\begin{array}{r|l} & 1\dot{4}44 \;|\; 38 \\ & 9 \\ 60 & 544 \\ & 480 \\ & 64 \\ & 64 = (8)^2 \\ & 0 \end{array}$$

Now, since the square of the tens of the root must be found exclusively in 14, the square root of the greatest perfect power in 14, which is 9, will be the tens of the root. 3, then, is the first figure of the root, and 30 will correspond to a in formula (1). Write the 3 to the right, as shown. Now, squaring it, subtract the result from 14, and bring down the 44. We have really subtracted $(30)^2=900$, and the remainder is what is left after taking away a^2 from the formula. It must then correspond to the $2ab+b^2$ of the formula. We should be able to find b at once from this remainder, if it were $2ab$ alone, by dividing by $2a$, (2×30), a being now known. But, at any rate, since b^2 is quite small compared with $2ab$, we shall not come far from b by dividing the remainder, as it stands, by $2a$. Then, doubling 30, and dividing the remainder by the result, we get 9. Now, multiplying 60 by 9 (giving $2ab$), and subtracting the result from the remainder, we shall have 4 left. But this second remainder must be equal to the square of the units, b^2; in this case $(9)^2=81$. If our original number, then, is a perfect power, 9 is too great.

Trying 8 we find 64 for a remainder, and this is just equal to $(8)^2$: 38, then, is the root required.

To shorten the last part of the operation we may double the 3, and divide the remainder, exclusive of the last figure, by it; then write the result in the unit's place, after 6. Now, when we multiply 68 by 8, we form at once $(2a+b)b=2ab+b^2$.

```
 1444|38
 9
68|544
   544
     0
```

When the number requires more than two places in the root, we might apply the same reasoning to the discovery of the two figures of the root of the highest denomination, and then, regarding all these as forming one denomination, proceed as before. This is accomplished by simply pointing off from the right, in places of two figures each, and continuing the operation as above. If we had two more places of figures in the above example, we should bring them down, and then double 38, and divide the remainder, exclusive of the right hand figures, by the result, writing it in the root, and also in the new trial division, and so proceed as before.

```
    15129|123
    1
 22|51
    44
243| 729
     729
       0
```

70. Extraction of the *n*th Root of Numbers.

Let it be required to find any root of a number, N. The root will consist of a tens and b units. Thus we shall have,

$$N=(a+b)^n=a^n+na^{n-1}b+, \text{ etc.}$$

Now the nth power of the tens must have 0's in the last n places, so that in looking for the tens of the root we may point off the last n places of the number. We may now find the tens of the root by finding the highest perfect root in the left hand period. Subtracting the nth power of the tens so found from the number, we shall have $na^{n-1}b+$, etc., left. Now, by forming n times the $n-1$ power of the tens, and dividing the remainder from it, we shall have the units of the root, or something too great. By raising the trial root so found to the nth power, we shall find whether it produces the number or not. If too great, the figure in the unit's place must be diminished.

We shall now give an example in cube root. Any other root may be found in like manner.

Let us find the cube root of 12977875.

It will be observed that only the first figure of the second period is brought down for a first remainder. This is because 12, the first divisor, is really 1200, so that the remaining two figures would be cut off in dividing, if brought down.

$$\begin{array}{rl}
 & 1\dot{2}97\dot{7}87\dot{5}\,|\,235 \\
 & 8 \\
3\times 2^2 = & 12\,|\,49 \\
 & 12977 \quad = \text{1st two periods.} \\
(23)^3 = & 12167 \\
3\times(23)^2 = & 1587\,|\,8108 \\
 & 12977875 = \text{The three periods.} \\
(235)^3 = & 12977875 \\
 & 0
\end{array}$$

EXAMPLES.

1. Find the square root of 651249. *Ans.* 807.
2. Find the cube root of 12167. *Ans.* 23.
3. Find the cube root of 421875. *Ans.* 75.
4. Find the square root of 1058841. *Ans.* 1029.
5. Find the cube root of 354894912. *Ans.* 708.
6. Find the fifth root of 248832. *Ans.* 12.

Remark.—Let it be remembered that $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$, and that, therefore, when the index is a multiple of two or more factors, we may find the roots indicated by those factors, successively, instead of extracting the entire root at once.

7. Find the sixth root of 244140625. *Ans.* 25.
8. Find the eighth root of 214358881. *Ans.* 11.

Remark.—If the number is partly or altogether *decimal*, begin to point off from the decimal point going to the left for the entire part and to the right for the decimal. Any number of 0's may be added to the decimal and the operation thus continued to any degree of approximation.

9. Find the square root of 657.4096. *Ans.* 25.64.
10. Find the square root of .140625. *Ans.* .375.

Remark.—A common fraction may be converted into a decimal and the root extracted to any desired degree of accuracy.

11. Find the square root of $\frac{39}{7}$. *Ans.* 2.3604.

12. Find the square root of $\frac{1}{3}$ to five places of decimals.

Ans. .57735.

13. Find the square root of 7, to four places of decimals.

Ans. 2.6457.

71. No Exact Root of an Imperfect Power.

The $\sqrt{5}$, or $\sqrt[3]{7}$, or the root of any imperfect power of the degree indicated, cannot be found exactly, either in entire or fractional quantities; but we may approximate such a root as nearly as we please: for example, $\sqrt{7}=2.64575131+$ and so on indefinitely.

Such quantities are said to be *incommeasurable*, and are commonly called *surds*.

The distinction between an imaginary quantity and a surd is, that we may obtain the root of a surd as nearly as we please without being able to find it exactly, while we cannot make the first movement towards finding the root of an imaginary expression.

Let us now prove that we cannot find the exact root of a surd.

Let p be any whole number whose nth root is $\frac{a}{b}$, a fraction having no common factor in the numerator and denominator. $\frac{a}{b}$ cannot, then, be reduced to an entire quantity. We shall have,

$$\sqrt[n]{p}=\frac{a}{b}.$$

Raising both numbers to the nth power, since they must still be equal, we shall have

$$p=\frac{a^n}{b^n}.$$

Now, in raising any number to a power, we but repeat the factors composing it, a certain number of times, introducing no new ones, so that a^n and b^n are still *prime* with respect to each other; that is, they have no common factor, and thus, $\frac{a^n}{b^n}$ is an irreducible fraction; so that we have p, a whole number, equal to an irreducible fraction, which is impossible. Then, since $\sqrt[n]{p}$ cannot be a whole number nor a fraction, it cannot be found exactly at all.

SECTION VII.

EQUATIONS.

72. *An Equation* is the indicated equality of two algebraic expressions; thus,

$$ax+by=c+d$$

is an equation.

The sign $=$ divides the equation into two parts, called *Members*: the part written first being the *First Member*, and that following the sign of equality, the *Second Member*.

73. The essence of an equation (its life, so to speak) lies in the equality of its members. Any operation may be performed upon it, so long as the equality is preserved intact.

The following are self-evident truths, called *Axioms*:

1. If equal quantities be added to both members of an equation, the equality will still subsist.
2. If equal quantities be subtracted from both members, the equality will still subsist.
3. If both members be multiplied by the same quantity, the equality will still subsist.
4. If both members be divided by the same quantity, the equality will still subsist.
5. If both members be raised to the same power, the equality will still subsist.
6. If the same root of both members be extracted, the equality will still subsist.

All operations upon equations are founded upon these axioms.

74. The First Transformation.

There are four principal transformations to which equations are submitted. We shall consider them in their order.

I. The object of the *First Transformation* of equations is to clear an equation of fractional quantities.

Take the equation,

$$\frac{x}{a}-\frac{y}{b}=\frac{c}{d}+f.$$

Now, we may multiply both members by any quantity, and still preserve the equality. Let us, then, multiply each term by abd, the common multiple of all the denominators. We shall have,

$$\frac{abdx}{a}-\frac{abdy}{b}=\frac{abdc}{d}+abdf.$$

There is now a common factor in the numerator and denominator of each fraction. Striking them out, we have,

$$bdx-ady=abc+abdf,$$

an equation in which all the terms are entire.

It is generally more convenient to strike out the common factor from the multiple before multiplying by it. We may thus say, that to clear an equation of fractions, or,

To make the first transformation of an equation, form the least common multiple of all the denominators; divide this by each denominator in succession and multiply each term respectively by the result. Entire terms are to be multiplied by the common multiple as it stands.

It is sometimes better to multiply each numerator by all the denominators except its own, and the entire terms by all the denominators.

EXAMPLES.

Clear the following equations of fractions.

1. $\frac{x}{2}-\frac{a}{b}+c=\frac{y}{4}-1.$

Explanation.—Here the least common multiple is $4b$. Dividing it by 2 and multiplying the numerator of the first term by $2b$, we have $2bx$ for the first term of the result. The other terms are found in like manner.

Ans. $2bx-4a+3bc=by-4b.$

2. $\frac{ax}{5}+\frac{c}{10a}=y+\frac{x}{5}.$ *Ans.* $2a^2x+c=10ay+2ax.$

3. $\frac{x-a}{4x}-\frac{x-1}{12}=a.$ *Ans.* $3x-3a-x^2+x=12ax.$

4. $\frac{1}{a+x}-1=\frac{a+x}{a-x}.$ *Ans.* $a-x-a^2+x^2=(a+x)^2.$

5. $\frac{\sqrt{x}}{a}-\frac{\sqrt{x}+1}{a^2}=\frac{5b}{c}.$ *Ans.* $ac\sqrt{x}-c\sqrt{x}-c=5a^2b.$

6. $\dfrac{\sqrt{x+y}}{a}=1-\dfrac{1}{\sqrt{x+y}}.$ *Ans.* $x+y=a\sqrt{x+y}-a.$

7. $\dfrac{x^{\frac{1}{2}}}{6a^2}-\dfrac{y^{-1}}{12b^2}=a^{-1}.$ *Ans.* $2b^2x^{\frac{1}{2}}-a^2y^{-1}=12ab^2.$

75. The Second Transformation.

II. *The object of the Second Transformation of equations is to transpose terms from one member of an equation to the other.*

Take the equation,

$$ax-cd=2by+a.$$

Now, as we may add the same quantity to both members without affecting the equality, let us add $-2by$ to both members. We shall have,

$$ax-cd-2by=2by+a-2by.$$

In the second member we now have two quantities equal with contrary signs, which we may cancel, and we shall thus have,

$$ax-cd-2by=a.$$

It will be observed that $2by$ has disappeared from the second member, and appeared in the first with its sign changed.

Now suppose we desired to move the term $-cd$ from the first member. Adding cd to both members, and cancelling as before, we have,

$$ax-2by=a+cd.$$

The term has disappeared from the first and appeared in the second with a contrary sign. Thus,

We may transpose any term from one member of an equation to the other by changing its sign.

Examples.

Transpose all terms containing an unknown quantity to the first member and all others to the second.

1. $ax-b=3x^2-y.$ *Ans.* $ax-3x^2+y=b.$

2. $a+25=-x^2+y.$ *Ans.* $x^2-y=-a-25.$

3. $21a^2b-xy-1=x.$ *Ans.* $-xy-x=1-21a^2b.$

4. $\frac{a^2}{b}-1=-\frac{x}{2}-4.$ *Ans.* $\frac{x}{2}=1-\frac{a^2}{b}-4.$

Remark.—Equations are usually cleared of fractions before transposing, but transpositions may be made at any time.

Transpose all the terms of the following to the first member:

5. $ax^2-1=bx-cx^3.$ *Ans.* $cx^3+ax^2-bx-1=0.$

6. $0=a+\frac{a}{b}y-\frac{1}{2}y^2+y^3-y^4.$ *Ans.* $y^4-y^3+\frac{1}{2}y^2-\frac{a}{b}y-a=0.$

7. $\sqrt{a+b}=-(a-b)x^2+\sqrt{a-b}\cdot y.$

Ans. $(a-b)x^2-\sqrt{a-b}\cdot y+\sqrt{a+b}=0.$

8. $1=\frac{x}{a-b}-\frac{a+b}{x}.$ *Ans.* $\frac{a+b}{x}-\frac{x}{a-b}+1=0.$

Remark.—If there are indicated products in an equation, it is generally better to develop such expressions before transposing.

In general, *transpose unknown terms to the first member, and known terms to the second.*

9. $\frac{(a+b)^2x}{c}-c=\frac{(x-1)\times 3}{5}.$

$$5(a+b)^2x-5c^2=(x-1)\times 3c.$$
$$5a^2x+10abx+5b^2x-5c^2=3cx-3c.$$

Ans. $5a^2x+10abx+5b^2x-3cx=5c^2-3c.$

76. The Third Transformation.

III. The object of the *Third Transformation* of equations is to so change an equation that the several powers of the same unknown quantity shall enter it but once.

Let us take an equation with several powers of the same unknown quantity; thus,

$$3ax^2-bx+x^3-2x+x^2-ax^3=1.$$

Writing the terms containing the same power of the unknown quantity together, and factoring with respect to these several powers, we have,

$$(1-a)x^3+(3a+1)x^2-(b+1)x^2=1.$$

Or taking a numerical equation,

$$5x^2-2x^2+10x-x=5;$$

Simplifying, we have,

$$3x^2+9x=5.$$

If an equation has any number of unknown quantities, the same course may be pursued.

Practically to effect the third transformation,

Gather together the terms containing the like powers of the same unknown quantity or quantities, and factor with respect to those several powers.

EXAMPLES.

Submit the following to the Third Transformation, transposing unknown terms to the first member, and known to the second.

1. $ax-b+cx^2-x+1=2x^2.$ *Ans.* $(c-2)x^2-(1-a)x=b-1.$

Caution.—Be careful in putting on or taking off a parenthesis with the − sign before it.

2. $1=a-\sqrt{2}\cdot x^2-ax+dx^2-x^2+\sqrt{a}\cdot x.$

Ans. $(1+\sqrt{2}-d)x^2-(\sqrt{a}-a)x=a-1.$

3. $3x^2-2x+1=a^2x^2-b^3x+cx+f.$

Ans. $(3-a^2)x^2+(b^3-c-2)x=f=1.$

4. $a^{\frac{1}{2}}x^{\frac{1}{2}}-cx=ax^{\frac{1}{2}}-2x+25.$ *Ans.* $(2-c)x+(a^{\frac{1}{2}}-a)x^{\frac{1}{2}}=25.$

5. $3x^2-2x+1=7x^2-4x.$ *Ans.* $-4x^2+2x=-1.$

77. Fourth Transformation.

IV. The object of the *Fourth Transformation* of equations is to make the co-efficient of the highest power of the unknown quantity unity.

Let us take an equation upon which the first three transformations have been already performed; thus,

$$(3a-b)x^2-(4+c)x=1.$$

We may divide both members, that is to say every term, by the co-efficient of x^2, and we shall have,

$$x^2-\frac{4+c}{3a-b}x=\frac{1}{3a-b}.$$

The same course may be pursued in any case; hence, having performed the previous transformations, to make the Fourth,

Divide each term by the co-efficient of the highest power of the unknown quantity.

EXAMPLES.

Apply the Fourth Transformation to the following:

1. $(a+b)x^2-x=b$. *Ans.* $x^2-\dfrac{x}{a+b}=\dfrac{b}{a+b}$.

2. $(3-c^2)x^3-(a^2+b)x^2+2x=4d^3-c$.

Ans. $x^3-\dfrac{a^2+b}{3-c^2}x^2+\dfrac{2}{3-c^2}x=\dfrac{4d^3-c}{3-c^2}$.

3. $(2-d^2)x=a^{\frac{1}{2}}-b^{\frac{1}{2}}$. *Ans.* $x=\dfrac{a^{\frac{1}{2}}-b^{\frac{1}{2}}}{2-d^2}$.

4. $25x=40$. *Ans.* $x=\dfrac{40}{25}$.

5. $(a-b)^n x=c^n$. *Ans.* $x=\dfrac{c^n}{(a-b)^m}$.

Perform all four of the transformations in succession on the following; transposing unknown terms to the first member and known to the second.

7. $\dfrac{ax}{b}-d=\dfrac{bx}{c}+1$. *Ans.* $x=\dfrac{bc+bcd}{ac-b^2}$.

8. $-\dfrac{a^2+b^2}{c}-\dfrac{c-d}{c^2}x=x+\dfrac{a}{b}$. *Ans.* $x=\dfrac{ac^2+a^2bc+b^3c}{bd-bc-bc^2}$.

9. $\dfrac{x}{2}-\dfrac{1}{2}+25=\dfrac{x}{4}-1$. *Ans.* $x=-102$.

10. $\dfrac{25}{3}-\dfrac{5x}{6}=1-\dfrac{10x}{12}-x$. *Ans.* $x=-\dfrac{88}{12}$.

78. The Change of Signs.

Let us have the equation,

$$-ax+by=-c.$$

Multiplying both members by -1, we have,

$$ax-by=c,$$

an equation in which all the signs have undergone a change. Hence, *we may change the signs of every term of an equation, and the equality will still subsist.*

79. Solution of Simple Equations.

An equation containing but the first power of the unknown quantity or quantities is called a simple *equation*, or an equation of the *first degree.*

It will be observed, that when we have submitted a simple equation containing a single unknown quantity to the four transformations in succession, the unknown quantity is made to stand alone in the first member, equal to known quantities in the second. We have thus found the value of the unknown quantity, and are said to have *solved* the equation.

Then, to solve a simple equation containing but one unknown quantity, we have but to submit it to all four of the transformations in succession. These four steps, in few words, are,

1. *Clear the equation of fractions ;*
2. *Transpose unknown terms to the first member and known to the second ;*
3. *Factor the first member with respect to the unknown quantity ;*
4. *Divide both members by the co-efficient of the unknown quantity.*

The value of the unknown quantity so found is called the *Root* of the equation. The root of an equation, then, is such a value of the unknown quantity as will *verify* the equation ; that is, when substituted for that quantity in the equation it will show that the two members are identical ; thus, the root of

$$\frac{ax}{b}-\frac{-b}{2}\cdot x=1$$

is

$$x=\frac{2b}{2a+b^2}.$$

Writing this in the equation, we have,

$$\frac{a\left(\frac{2b}{2a+b^2}\right)}{b}-\frac{-b}{2}\left(\frac{2b}{2a+b^2}\right)=1.$$

Simplifying and clearing of fractions, we have,

$$4ab+2b^3=4ab+2b^3.$$

Thus, the two members are entirely the same; and the equation must be true.

An equation may be made to have any two equal quantities in the two members; thus, cancelling equal terms in both members, we have,

$$0=0.$$

By dividing each member by itself, any equation will be

$$1=1.$$

The verification of a numerical equation is more simple; thus, solving

$$\frac{3x}{5}-\frac{x}{10}=25,$$

we have $x=50$.

This for x in the equation gives,

$$\frac{3(50)}{5}-\frac{50}{10}=25$$

$$300-50=250$$

$$250=250.$$

Examples.

1. Solve, $\frac{21-3x}{3}-\frac{4x+6}{9}=6-\frac{5x+1}{4}$. *Ans.* $x=3$.

2. $\frac{x-1}{2}+\frac{x+1}{2}=3x-12$. *Ans.* $x=6$.

3. $\frac{ax-b}{4}+\frac{a}{3}=\frac{bx}{2}-\frac{bx-a}{3}$. *Ans.* $x=\frac{3b}{3a-2b}$.

4. $2x-\frac{4x-2}{5}=\frac{3x-1}{2}$. *Ans.* $x=3$.

5. $a^2(x-1)+ab(x-2)=b^2$. *Ans.* $x=\frac{a+b}{a}$.

6. $\frac{\sqrt{a}\cdot x}{\sqrt{b}}-\frac{\sqrt{c}}{2}=x$. *Ans.* $x=\frac{\sqrt{bc}}{2(\sqrt{a}-\sqrt{b})}$.

7. $\dfrac{a^{\frac{1}{2}}-x}{c^{\frac{1}{2}}}-\dfrac{-x+1}{2}=2.$ *Ans.* $x=\dfrac{5c^{\frac{1}{2}}-2a^{\frac{1}{2}}}{c^{\frac{1}{2}}-2}.$

8. $\dfrac{7+9x}{4}-\left(1-\dfrac{2-x}{9}\right)=7x.$ *Ans.* $x=\dfrac{1}{5}.$

80. Degree of Equations.

The degree of an equation is determined by the greatest number of times the unknown quantity or quantities enter any one term as a factor; thus,

$$\left.\begin{array}{r}5x-7x=3\\ a^2x+by=4z-c\end{array}\right\}\text{ are of the first degree.}$$

$$\left.\begin{array}{r}5x^2-7x=3\\ a^2xy+bx=4z-c\end{array}\right\}\text{ are of the second degree.}$$

$$\left.\begin{array}{r}5x^3-7x=3\\ a^2xyz+bxy=4z-c\end{array}\right\}\text{ are of the third degree.}$$

81. Complete and Incomplete Equations.

A *complete* equation of any degree is one which contains all the several powers of the unknown quantity from that which determines its degree down to the zero power; thus,

$$ax^3-bx^2+cx+dx^\circ=0$$

is a complete equation of the third degree of one unknown quantity. x° being *unity*, we may write it or not as we please.

When one or more of the intermediate powers are wanting, the equation is said to be *incomplete;* thus,

$$\left.\begin{array}{l}x^3+bx^2=d\\ x^3-5x=d\\ x^3=d\end{array}\right\}\text{ are incomplete.}$$

82. Complete Equations of the Second Degree.

When an equation has undergone the four transformations already given, it is said to have its *Simplest Form.*

Let us take a complete equation of the second degree containing but one unknown quantity; thus,

$$\frac{x^2}{a}-\frac{x}{2}+c=d-\frac{2x}{3}+\frac{x^2}{b}.$$

Submitting it to the four transformations, we have,

$$x^2+\frac{ab}{6(b-a)}\cdot x=\frac{ab(d-c)}{b-a}.$$

Now, the co-efficient of x is a known quantity, and we may let $2p$ represent it. So we may let q be equal to the second member. Using these values in the equation, we have,

$$x^2+2px=q \quad - \quad - \quad (1).$$

This is called the *Reduced Form* of a complete equation of the second degree containing but one unknown quantity. The terms, however, may have different signs.

In any case, after making the sign of the first term plus, if not so already, the other two terms, $2px$ and q, must present one of the four following phases:

Both plus;
$2px$ minus and q plus;
$2px$ plus and q minus;
Both minus.

The equation (1) written in every possible form gives the following:

$$x^2+2px=\ \ q.$$
$$x^2-2px=\ \ q.$$
$$x^2+2px=-q.$$
$$x^2-2px=-q.$$

The application of the four transformations will manifestly bring any equation of this character to the form of one of these equations. They are thus called the *Four Forms* of a complete equation of the second degree.

83. Solution of Incomplete Equations.

If the equation is incomplete, there will be no term containing the first power of the unknown quantity; that is, $2p=0$, and the four forms will become,

$$x^2=+q.$$
$$x^2=-q.$$

We may extract the square root of both members of these equations, and shall have,

$$x=\pm\sqrt{q}.$$
$$x=\pm\sqrt{-q}.$$

It will be observed that the values of x in the last case are imaginary.

We have thus found the roots of the incomplete equations; and so in any case,

To solve an incomplete equation of the second degree, reduce it to the form of $x^2=q$, *and then extract the square root of both members.*

Reduce the following equations to the form of $x^2+2px=q$.

1. $\frac{a^2x^2}{b^2}-\frac{2ax}{c}+\frac{b^2}{c^2}=0.$ *Ans.* $x^2-\frac{2ab^2c}{a^2c^2}x=-\frac{b^4}{a^2c^2}$

2. $\frac{x+12}{x}+\frac{x}{x+12}=\frac{26}{5}.$ *Ans.* $x^2+12x=45.$

3. $\frac{ax}{b}+\frac{3x^2}{4}+1=\frac{1+b}{b}-\frac{x^2}{4}+\frac{x}{a}.$ *Ans.* $x^2+\frac{a^2-b}{ab}x=\frac{1}{b}.$

Reduce the following equations to the form of $x^2=q$, and solve.

4. $ax^2+\frac{b}{c}=-cx^2+1.$ *Ans.* $x=\pm\sqrt{\frac{c-b}{ac+c^2}}.$

5. $\frac{3x^2-2}{5}=4x^2-14.$ *Ans.* $x=\pm 2.$

6. $\sqrt{\frac{x^2-3}{2}}=\sqrt{\frac{9+3x^2}{12}}.$ *Ans.* $x=\pm 3.$

7. $(a+x)(a-x)=\frac{b}{c}-2x^2.$ *Ans.* $x=\pm\sqrt{\frac{b-a^2c}{c}}.$

8. $x\left(\frac{x}{2}-x\right)=6-\frac{2x^2}{3}.$ *Ans.* $x=\pm 6.$

9. $ax(b-x)^{\frac{1}{2}}=a(1-x^3)^{\frac{1}{2}}.$ *Ans.* $x=\pm\sqrt{\frac{1}{b}}.$

84. Solution of Complete Equations.

Resuming the equation,

$$x^2+2px=q \cdot \cdot \cdot \cdot (1),$$

we see that the first member may be made a perfect square by adding p^2 to it. This we are at liberty to do, provided that the same quantity be added to the second member also. We may thus have,

$$x^2+2px+p^2=q+p^2.$$

Extracting the square root of both members, we have,

$$x+p=\pm\sqrt{q+p^2}.$$

Transposing p to the second member,

$$x=-p\pm\sqrt{q+p^2} \quad . \; . \; . \; . \; . \quad (2).$$

These, then, are the values of x in equation (1). There are always two roots; written separately, they are,

$$x=-p+\sqrt{q+p^2} \text{ and } x=-p-\sqrt{q+p^2}.$$

It will be observed that the entire part in each root, $-p$, is altogether the same, and is half the co-efficient of the first power of x in the equation (1) from which they came, taken with a contrary sign; the radical parts are also the same, but one of them has the plus, and the other the minus sign.

It is obvious that we may at once write out the roots of such an equation, after it has been reduced to this trinomial form, by substituting the values of p and q, taken from the equation itself, in these roots.

For example, let us have,

$$ax^2-\frac{c}{2}x=7-x^2.$$

The application of the four transformations gives us,

$$x^2-\frac{c}{2a+2}\cdot x=\frac{14}{2a+2}.$$

Now, in this equation, p, the half co-efficient of x, is $\frac{c}{4a+4}$; and q is $\frac{14}{2a+2}$.

These, in the formulas for the roots (2) give us

$$x=\frac{c}{4a+4}+\sqrt{\frac{14}{2a+2}+\left(\frac{c}{4a+4}\right)^2}, \text{ and}$$

$$x=\frac{c}{4a+4}-\sqrt{\frac{14}{2a+2}+\left(\frac{c}{4a+4}\right)^2}.$$

We may solve any such equation in the same manner; so that we can say, in general,

To solve a complete equation of the second degree, containing one unknown quantity,

1. *Apply the four transformations in succession, thus reducing it to the form* $x^2+2px=q$.

2. *Write the unknown quantity equal to half the co-efficient of the first power of that quantity, with a contrary sign, plus and minus the square root of the second member, augmented by the square of this half co-efficient.*

EXAMPLES.

1. Solve $\frac{10}{x}-\frac{14-2x}{x^2}=\frac{22}{9}$.

By the four transformations,

$$x^2-\frac{108}{22}x=-\frac{126}{22}.$$

Writing out the roots,

$$x=\frac{54}{22}\pm\sqrt{-\frac{126}{22}+\left(\frac{54}{22}\right)^2}.$$

Performing the operations under the radical sign, as indicated,

$$x=\frac{54}{22}\pm\sqrt{\frac{144}{(22)^2}}.$$

Extracting the root,

$$x=\frac{54}{22}\pm\frac{12}{22}.$$

Whence,

$$x=3 \text{ and } x=\frac{21}{11}. \quad \textit{Ans.}$$

2. Solve $\frac{2x^2}{a}+\frac{bx}{a^2}=\frac{x}{b}+\frac{x^2}{a}+\frac{1}{a}$.

$$x^2-\frac{a^2-b^2}{ab}x=1.$$

$$x=\frac{a^2-b^2}{2ab}\pm\sqrt{1+\left(\frac{a^2-b^2}{2ab}\right)^2}.$$

$$x=\frac{a^2-b^2}{2ab}\pm\sqrt{\frac{4a^2b^2}{4a^2b^2}+\frac{a^2-2a^2b^2+b^4}{4a^2b^2}}.$$

$$x=\frac{a^2-b^2}{2ab}\pm\sqrt{\frac{a^4+2a^2b^2+b^4}{4a^2b^2}}.$$

$$x=\frac{a^2-b^2}{2ab}\pm\frac{a^2+b^2}{2ab}.$$

$$x=\frac{a}{b} \text{ and } x=-\frac{b}{a}. \quad \textit{Ans.}$$

3. $3x^2-2x=65$. *Ans.* $x=5$, $x=-4\frac{1}{3}$.

4. $ax^2-b=bx-\frac{1}{b^2}$. *Ans.* $x=\frac{b}{2a}\pm\frac{\sqrt{4ab^3-4a+b^4}}{2ab}$.

5. $\frac{x}{x+60}=\frac{7}{3x-5}$. *Ans.* $x=14$, $x=-10$.

6. $\frac{a^2x^2}{b^2}-\frac{2ax}{c}+\frac{b^2}{c^2}=0$. *Ans.* $x=\frac{b^2}{ac}\pm 0$.

7. $\frac{48}{x+3}=\frac{165}{x+10}-5$. *Ans.* $x=\frac{27}{5}$, $x=5$.

8. $\frac{x+12}{x}+\frac{x}{x+12}=\frac{26}{5}$. *Ans.* $x=3$, $x=-15$.

9. $\frac{ax}{b}+\frac{3x^2}{4}+1=\frac{1+b}{b}-\frac{x^2}{4}+\frac{x}{a}$. *Ans.* $x=\frac{1}{a}$, $x=-\frac{a}{b}$.

10. $\frac{2x-3}{3x-5}+\frac{3x-5}{2x-3}=\frac{5}{2}$. *Ans.* $x=\frac{7}{4}$, $x=1$.

85. Trinomial Equations.

A *trinomial equation* is one which contains but two different powers of the unknown quantity. Such equations, when simplified, have the form,

$$x^m+2px^n=q.$$

Complete equations of the second degree, such as we have just been considering, are a particular case of this general form, m being 2 and n unity.

The method just explained, of solving a complete equation of the second degree, is equally applicable for the solution of any trinomial equation, when $m=2n$; that is, when the equation has the form,

$$x^{2n}+2px^n=q.$$

We can make the first member a perfect square in the same manner as in the last article, and we should have, after extracting the square root of both members and transposing,

$$x^n = -p \pm \sqrt{q+p^2}.$$

Now extracting the nth root of both members, we have,

$$x = \sqrt[n]{-p \pm \sqrt{q+p^2}}.$$

To solve such an equation, then, we adopt the same method as in the case of an equation of the second degree, except that we write the unknown quantity with an exponent equal to half of its highest exponent in the reduced equation, equal to the roots, and after that extract the nth root of both numbers of these new equations.

For example:

$$x^4 - 4x^2 = 12,$$

whence, $$x^2 = 2 \pm \sqrt{12+4},$$

and thus, $$x = \pm \sqrt{2 \pm \sqrt{16}};$$

or, $$x = \pm \sqrt{2 \pm 4}.$$

It will be observed that there are four roots in this equation. Written separately, they are,

$$x = +\sqrt{6}, \quad x = -\sqrt{6}, \quad x = +\sqrt{-2}, \text{ and } x = -\sqrt{-2}.$$

It will be observed, further, that two of them are imaginary.

It may be well to remark here, that every such equation has as many roots as there are units in the number which indicates its degree. Sometimes some of the roots are equal to each other and some are imaginary.

EXAMPLES.

1. Solve, $x^{2n} - 2x^n = 8$. *Ans.* $x = \sqrt[n]{4}$, $x = \sqrt[n]{-2}$.

Remark.—Where only two roots are given in the answers, the student may find the others.

2. $x^4 + x^2 = 20$. *Ans.* $x = \pm 2$, $x = \pm \sqrt{-5}$.

3. $x^6 + 4x^3 = 96$. *Ans.* $x = 2$, $x = \sqrt[3]{-12}$.

4. $(x+y)^2 - (x+y) = 6$. *Ans.* $x+y = 3$, $x+y = -2$.

Remark.—Here $x+y$ is to be regarded as a single quantity.

5. $(x-1)^2-(x-1)=6$. *Ans.* $x=4$, $x=-1$.

6. $x-\sqrt{x}=6$. *Ans.* $\sqrt{x}=3$, $\sqrt{x}=-2$.

7. $\sqrt{x}-\sqrt[4]{x}=6$. *Ans.* $\sqrt[4]{x}=3$, $\sqrt[4]{x}=-2$.

86. Simultaneous Equations.

Simultaneous Equations are those in which any quantity in one of the equations is the same in quantity and quality as that quantity in any other of the equations; thus, if

$$5x+3y=10, \text{ and}$$
$$3x-7y=15,$$

are simultaneous equations, and x in the one has a particular value, as 5, it must have the same value in the other; and again, if it stand for a particular kind of quantity, as pounds, in the one, it must be pounds in the other. So, also, with y, or any other unknown quantities which may enter such equations.

If the known quantities are represented by letters, the same thing is true of them as well; thus, if

$$ax+by=c$$
$$dx-ay=b$$

are simultaneous, then a must have the same numerical value and represent the same kind of quantity in both equations.

It is plain that only simultaneous equations can be combined, as, for example, added member to member, or multiplied member by member, for if the same symbols should represent different things in the several equations, the result of such combination, though a true equality, would mean nothing.

Equations are commonly combined for the purpose of getting rid of one or more of the unknown quantities which enter them; or, as it is said, in order to *eliminate* such unknown quantities.

There are three methods, commonly in use, of combining equations for the purpose of eliminating unknown quantities. They are,

1*st.* *By Addition.*
2*d.* *By Substitution.*
3*d.* *By Comparison.*

In general, the first thing to be done with equations preparatory to combining them by any one of these methods, is to subject them to the first three transformations. The same unknown quantity will then enter any one of the equations but once, if the equations are of the first degree. If of a higher degree, any particular power of the same unknown quantity will enter but once.

87. Elimination by Addition.

Let us now consider these three methods of elimination in the order given.

Take two simultaneous equations of the first degree, one numerical and the other literal, in order to make the case more general, and suppose them already brought to the proper form; thus,

$$5x-3y=10$$
$$ax+by=c.$$

Now, we may multiply the first equation through by a, and the second by 5, without disturbing the equality in either case. We shall thus have,

$$5ax-3ay=10a.$$
$$5ax+5by=5c.$$

We may further change the signs of all the terms of either of the equations, and still preserve the equality. Changing the signs of the second, we have,*

$$5ax-3ay=10a$$
$$\underset{-}{5ax}\underset{-}{+}5by=\underset{-}{5c}.$$

Now, adding the equations member by member, we have,

$$-3ay-5by=-5c+10a.$$

and from this,

$$y=\frac{5c-10a}{3a+5b}.$$

In like manner, we may find the value of x; or, we may substitute this value of y in either one of the equations, and find it in that way; thus, writing this value of y in the first of the given equations, we have,

$$5x-3\left(\frac{5c-10a}{3a+5b}\right)=10.$$

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* It will be found best not to destroy the original signs, but simply to write the new ones below, and a little to the right of the old ones, as here shown.

Solving,

$$x=\frac{10b+3c}{3a+5b}.$$

It will be observed that the object in this method of elimination is to make the terms containing the particular unknown quantity which we want to be rid of, cancel when we come to add the equations. These terms must be made entirely the same and must have different signs, in order to do this. We multiply or divide the two equations by any quantities which will cause the terms in question to become alike, and if they have not already different signs, we change all the signs of one of the equations to make them differ.

We may say, then, practically,

To eliminate a quantity by addition,

1. *Subject the equations, if necessary, to the first three transformations;*

2. *Multiply or divide either or both equations by whatever will make the terms to be cancelled the same;*

3. *Make the signs of these terms differ, if they do not already, by changing all the signs in one of the equations, and then add member to member.*

The resulting equation will be free from the quantity in question.

EXAMPLES.

1. Solve $\frac{4x}{2}-5=\frac{x+2y}{4}.$

$$7x+4y=\frac{24-6x}{6}.$$

Simplifying, we have,

$$7x-\ 2y=20$$
$$48x+24y=24.$$

Dividing the second equation by 12,

$$7x-2y=20$$
$$4x+2y=2.$$

Adding and then solving,

$$x=2.$$

This value of x in any of the above equations gives,

$$y=-3.$$

2. Solve $\begin{cases} 3x+5y=21 \\ 4y-2x=8. \end{cases}$ *Ans.* $x=2$, $y=3$.

3. Solve $\begin{cases} ax=by \\ x=c-y. \end{cases}$ *Ans.* $x=\dfrac{bc}{a+b}$, $y=\dfrac{ac}{a+b}$.

4. Solve $\begin{cases} 5x-8\frac{1}{2}=7y-44 \\ 2x=y+\frac{5}{7}. \end{cases}$ *Ans.* $x=4\frac{1}{2}$, $y=8\frac{2}{7}$.

5. Solve $\begin{cases} ax+by=c \\ fx+gy=h. \end{cases}$ *Ans.* $x=\dfrac{cg-bh}{ag-bf}$, $y=\dfrac{ah-cf}{ag-bf}$.

6. Solve $\begin{cases} 13x+7y-341=\dfrac{15y}{2}+\dfrac{67x}{2} \\ 2x+\dfrac{y}{2}=1. \end{cases}$ *Ans.* $x=-18\frac{18}{37}$, $y=75\frac{35}{37}$.

7. Solve $\begin{cases} (x+5)(y+7)=(x+1)(y-9)+112 \\ 2x+10=3y+1. \end{cases}$ *Ans.* $x=3$, $y=5$.

8. Solve $\begin{cases} 3x+5y=\dfrac{(8b-2a)ab}{b^2-a^2} \\ b^2x-\dfrac{a^2bc}{a+b}+(a+b+c)ay=a^2x(2a+b)ab. \end{cases}$

Ans. $x=\dfrac{ab}{b-a}$, $y=\dfrac{ab}{a+b}$.

88. Elimination by Substitution.

Resuming the equations in the last article,

$$5x-3y=10,$$
$$ax+by=c,$$

let us find the value of one of the unknown quantities, as x, in terms of the other in, say, the first equation; that is, let us regard all the quantities in the first equation as known, except x, and then find the value of x, under this hypothesis. We shall have,

$$x=\frac{10+3y}{5}.$$

Now, we may place this value of x for that quantity in the second equation; thus,

$$a\left(\frac{10+3y}{5}\right)+by=c.$$

We have now an equation with but one unknown quantity in it. Solving this, we have,

$$y=\frac{5c-10a}{3a+5b}.$$

Substituting this result for y in either of the equations, we have,

$$x=\frac{10b+3c}{3a+5b}.$$

This is the method of elimination by *Substitution.* We may thus say that,

To eliminate a quantity by Substitution, find the value of such quantity in terms of the remaining quantities from one of the equations, and substitute this value in the other.

The resulting equation will be free from the quantity in question.

EXAMPLES.

1. Solve $\begin{cases} \dfrac{x}{2}-1=2-\dfrac{y}{3} \\ 2x=\dfrac{2y}{3}+6. \end{cases}$ *Ans.* $x=4$, $y=3$.

2. Solve $\begin{cases} x=y-1 \\ x-\dfrac{y}{3}=2x-11. \end{cases}$ *Ans.* $x=8$, $y=9$.

3. Solve $\begin{cases} x+y=a \\ x-y=b. \end{cases}$ *Ans.* $x=\dfrac{a+b}{2}$, $y=\dfrac{a-b}{2}$.

4. Solve $\begin{cases} \dfrac{2x}{5}+\dfrac{3y}{4}=\dfrac{9}{20} \\ \dfrac{3x}{4}+\dfrac{2y}{5}=\dfrac{61}{120}. \end{cases}$ *Ans.* $x=\dfrac{1}{2}$, $y=\dfrac{1}{3}$.

5. Solve $\begin{cases} \dfrac{x}{a}+5=\dfrac{y}{2}-x \\ x+\dfrac{y}{2}=1-ay. \end{cases}$ *Ans.* $x=$ - - -, $y=$ - - -.

Let the examples in the previous article be solved by this method.

89. Elimination by Comparison.

Take, again, the equations

$$5x-3y=10$$
$$ax+by=c.$$

Find the value of x in both equations in terms of y; thus,

$$x=\frac{10+3y}{5}.$$

$$x=\frac{c-by}{a}.$$

These must be equal to each other, and thus we have,

$$\frac{10+3y}{5}=\frac{c-by}{a}.$$

Solving, we have, as before,

$$y=\frac{5c-10a}{3a+5b}.$$

Finding x in the same way or by substitution, we have,

$$x=\frac{10b+3c}{3a+5b}.$$

This is called the method of elimination by *Comparison*. Practically,

To eliminate a quantity by Comparison, find the value of the quantity in both equations in terms of the remaining quantities, and place these values equal to each other.

The resulting equation will be free from the quantity in question.

EXAMPLES.

1. Solve $\begin{cases} 16x+30y=18 \\ 12x-24y=-2. \end{cases}$ *Ans.* $x=\frac{1}{2}$, $y=\frac{1}{3}$.

2. Solve $\begin{cases} \frac{x}{7}=99-7y \\ 7x=51-\frac{y}{7}. \end{cases}$ *Ans.* $x=7$, $y=14$.

3. Solve $\begin{cases} ax+by=c \\ dx+fy=g. \end{cases}$ *Ans.* $x=\dfrac{fc-bg}{fa-bd}\cdot y=\dfrac{cd-ag}{bd-af}.$

4. Solve $\begin{cases} abx=\dfrac{y}{2} \\ aby=1-x. \end{cases}$ *Ans.* $x=\dfrac{1}{2a^2b^2+1}\cdot y=\dfrac{2ab}{2a^2b^2-1}.$

5. Solve $\begin{cases} x+y=18.73 \\ 0.56x+13.42y=763.4. \end{cases}$ *Ans.* $x=-39.81$
$y=\ \ 58.54.$

Let the examples in the last two articles be used for practice by this method.

90. Elimination in General.

Let us now extend the principle of elimination to the solution of equations containing any number of unknown quantities.

Let us have three equations which have been already subjected to the first three transformations, and containing three unknown quantities; thus,

$$2x+3y-z=1 \quad - \ - \ - \ - \quad (1).$$
$$3x-2y+3z=2 \quad - \ - \ - \ - \quad (2).$$
$$x+y-z=3 \quad - \ - \ - \ - \quad (3).$$

Manifestly we may eliminate one of the unknown quantities from any two of these equations, by either of the three methods already given. Let us, say, combine the first two, and get rid of z. Multiplying the first by 3, and adding, we have

$$9x+7y=5 \quad - \ - \ - \ - \quad (4).$$

In the same way, combining the first and last, eliminating z, we have

$$x+2y=-2 \quad - \ - \ - \ - \quad (5).$$

We now have two equations (4) and (5), with but two unknown quantities. These we may now combine by any one of the three methods, and shall have

$$x=\frac{24}{11} \text{ and } y=-\frac{23}{11}.$$

Substituting these in either of the three original equations, we have

$$z=-\frac{32}{11}.$$

We have thus solved the group of equations. The same course may be pursued with any number of equations, provided that there are as many equations as there are unknown quantities.

Thus we may say, that

To solve a group of simultaneous equations, there being as many equations as there are unknown quantities,

Combine the equations two and two, being careful not to unite the same two more than once, and always eliminating the same quantity, until there are as many resulting equations as there are unknown quantities remaining.

Combine these resulting equations in the same way. We shall at last have an equation containing a single unknown quantity, which may be solved. Complete the solution by substituting the value of the quantity so found in one of the resulting equations, with two unknown quantities in it, and so on.

Examples.

1. Solve $\begin{cases} x+y=10 \\ x+z=19 \\ y+z=23. \end{cases}$ *Ans.* $x=3$, $y=7$, $z=16$.

2. Solve $\begin{cases} y=41-\frac{x}{2} \\ x=\frac{41}{2}-\frac{z}{4} \\ y=34-\frac{z}{5}. \end{cases}$ *Ans.* $x=18$, $y=32$, $z=10$.

3. Solve $\begin{cases} x+y+z=30 \\ 8x+4y+2z=50 \\ 27x+9y+3z=64. \end{cases}$ *Ans.* $x=\frac{2}{3}$, $y=-7$, $z=36\frac{1}{3}$.

4. Solve $\begin{cases} x+y=a-z \\ by=cx \\ dz=fx. \end{cases}$ *Ans.* $x=\frac{abd}{bd+cd+bf}$, $y=\frac{acd}{bd+cd+bf}$, $z=\frac{abf}{bd+cd+bf}$.

5. Solve $\begin{cases} \frac{1}{x}+\frac{1}{y}=a \\ \frac{1}{x}+\frac{1}{z}=b \\ \frac{1}{y}+\frac{1}{z}=c. \end{cases}$

Ans. $x=\frac{2}{a+b-c}$,
$y=\frac{2}{a-b+c}$,
$z=\frac{2}{b+c-a}$.

91. Equations of Condition.

If we should have more unknown quantities than we have independent equations, we shall have more than one unknown quantity in the last resulting equation.

If we should have more equations than there are unknown quantities, we may eliminate *all* the unknown quantities, and thus have a resulting equation between *known* quantities; thus, let us have,

$$y=ax+b.$$
$$y=cx+d.$$
$$y=fx+g.$$

Placing the second members of the first two equal to each other, we have,

$$ax+b=cx+d \quad \therefore x=\frac{d-b}{a-c}.$$

Then combining the first equation with the third, eliminating y, we have,

$$ax+b=fx+g \quad \therefore$$
$$x=\frac{g-b}{a-f}.$$

Placing these two values of x equal to each other, we have $\frac{d-b}{a-c}=\frac{g-b}{a-f}$, an equation in which there is no unknown quantity.

Now, if this last result is not a true equality, all the equations from which it was derived cannot be true. They could not all be satisfied at the same time for the same set of values for the unknown quantities. Such a resulting equation between constants is called an *Equation of Condition;* since it is the condition upon which the equation from which it is derived can be true.

When there are more equations than there are unknown quantities, there must thus be a certain interdependence between the constants which enter them. The equations from which such equation of condition is derived, cannot, therefore, be independent equations.

Examples.

Find the equation of condition in the following groups of equations:

1. $\begin{cases} ax=b \\ cx=d \end{cases}$ *Ans.* $\frac{b}{a}=\frac{d}{c}$.

2. $\begin{cases} ax=by \\ cy=dx \\ fx=g. \end{cases}$ *Ans.* $\frac{dg}{cf}=\frac{ag}{bf}$.

3. $\begin{cases} y=ax+b \\ y=a'x+b. \end{cases}$ *Ans.* $a=a'$.

92. Simultaneous Equations of a Higher Degree.

Equations of a higher degree than the first can be combined and a single equation found containing a single unknown quantity. The method of elimination is altogether the same as that already explained; but the resulting equation is generally of too high a degree to be managed by methods falling within the province of this work: For example, let us have,

$$ax^2+by=c.$$
$$2y^2-3x=5.$$

Solving the first equation with respect to y, we have,

$$y=\frac{c-ax^2}{b}.$$

This substituted in the second, gives,

$$2\left(\frac{c-ax^2}{b}\right)^2-3x=5,$$

which is an equation of the fourth degree; and, in general, when both equations are of the second degree, the resulting equation will be of the fourth.

93. Simultaneous Equations of the First and Second Degrees.

There are, however, two classes of simultaneous equations, beyond the first degree, which admit of ready solution. They are,

1. *Equations containing two unknown quantities, when one is of the second degree and the other of the first.*

2. *When the equations are both of the second degree and homogeneous with respect to the unknown quantities.*

I. Let us take the first case. Assume,

$$6x^2-2xy=36$$
$$3x+y=12.$$

The value of y, in the second, in terms of x, is,

$$y=12-3x.$$

This substituted in the first gives,

$$6x^2-2x(12-3x)=36;$$

whence,
$$x^2-2x=3.$$
$$x=3 \text{ and } -1.$$

These values in the second equation give,

$$y=3 \text{ and } 15.$$

A like course may be pursued in any such case, so that we may always solve two simultaneous equations containing two unknown quantities, when one of them is of the first and the other of the second degree, by simply eliminating one of the unknown quantities.

94. Homogeneous Equations of the Second Degree.

II. When the equations are homogeneous with respect to the unknown quantities.

Take the equations,

$$x^2-2xy=1-3xy \quad - \quad - \quad (1),$$
$$y^2+4x^2-2=5x^2 \quad - \quad - \quad (2).$$

The solution is accomplished by using an auxiliary quantity. Applying the second and third transformations and making $y=px$, we have,

$$x^2+px^2=1. \quad \therefore\ x^2=\frac{1}{1+p} \quad - \quad - \quad (3),$$

$$p^2x^2-x^2=2. \quad \therefore\ x^2=\frac{2}{p^2-1} \quad - \quad - \quad (4).$$

Placing these two values of x^2 equal to each other,

$$\frac{1}{1+p}=\frac{2}{p^2-1}.$$

Whence,
$$p^2-2p=3$$
$$p=1\pm\sqrt{3+1}$$
$$p=3 \text{ and } -1.$$

Using the first value of p in either (3) or (4), we have,

$$x=\pm\tfrac{1}{2}.$$

The first values of p and x in $y=px$, give,

$$y=\tfrac{3}{2}.$$

The same course may be pursued with any two equations of this class.

Examples.

1. Solve $x^2+y^2=13$
   $x+y=5.$ *Ans.* $x=3,\ y=2.$

2. Solve $x^2+y^2=41$
   $x-y=-1.$ *Ans.* $x=4,\ y=5.$

3. Solve $x^2+y^2=13$
   $xy-y^2=2.$ *Ans.* $x=3,\ y=2.$

4. Solve $x^2+2xy+y^2=25$
   $3x-4y=\frac{x}{4}+\frac{1}{4}.$ *Ans.* $x=3,\ y=2.$

5. Solve $3x^2-2y^2=57$
   $\frac{3x+y}{2}=9.$ *Ans.* $x=5,\ y=3.$

6. Solve $3xy-2x^2=10$
   $2xy+2y^2=\frac{11x^2-y^2}{7}+25.$ *Ans.* $x=2,\ y=3.$

7. Solve $x+y=13$
   $\sqrt{x}+\sqrt{y}=5.$ *Ans.* $x=9,\ y=4.$

8. Solve $\sqrt{xy}+x=15$
   $\sqrt{x}-\sqrt{y}=1.$ *Ans.* $x=9,\ y=4.$

95. Solution of Simultaneous Equations of any degree.

The principles already established will enable us to solve many equations of the second and higher degrees by the exercise of a little ingenuity. No specific rules would prove of much service; so that the general management of such cases may be best exemplified by a few examples.

1. Solve $x^3-y^3=19$ - - (1),
   $x-y=1$ - - (2).

We may divide the first equation by the second, and thus have,

$$x^2+xy+y^2=19 \quad - \quad - \quad (3).$$

Now the value of x from the second, $x=1+y$, in equation (3), gives,

$$y^2+y=6.$$

Whence,
$$y=-\tfrac{1}{2}\pm\sqrt{6+\tfrac{1}{4}}$$
$$y=2,\ y=-3.$$

These values of y in (2) give,

$$x=-1,\ x=-2.$$

2. Solve $(x+y)^2+(x+y)=30 \quad - \quad - \quad (1),$

$$x^2-y^2=5 \quad - \quad - \quad (2).$$

From (1) we have,

$$x+y=-\tfrac{1}{2}\pm\sqrt{30+\tfrac{1}{4}},$$
$$x+y=-\tfrac{1}{2}\pm\tfrac{11}{2},\ x+y=5,\ x+y=-6.$$

Dividing (2) by these values, we have,

$$x-y=1,\ x-y=-\tfrac{5}{6}.$$

From these values of $x+y$ and $x-y$, we have,

$$x=3,\ x=-\tfrac{2}{3}.$$
$$y=2,\ y=\sqrt{-\tfrac{41}{9}}.$$

3. Solve $x^3-y^3=117 \quad - \quad - \quad (1),$

$$(x-y)^2-(x-y)=6 \quad - \quad - \quad (2).$$

From (2) we have

$$x-y=3,\ x-y=-2.$$

Dividing (1) by this first value of $x-y$ we have,

$$x^2+xy+y^2=39 \quad - \quad - \quad (3).$$

Substituting the value of x from $x-y=3$, in (3) and solving, we have,

$$y=2,\ y=-5,$$
$$x=5,\ x=-2.$$

4. Solve $x^3+y^3=152 \quad - \quad - \quad (1),$

$$x+y=8 \quad - \quad - \quad - \quad (2).$$

Dividing (1) by (2), we have,

$$x^2-xy+y^2=19.$$

Combining this with (2), we have,

$$x=5,\ x=3.$$

Whence, $$y=3,\ y=5.$$

5. Solve $x^2+y^2=74$ - - (1),

$xy\ =35$ - - (2).

Multiplying (2) by 2, we have,

$$2xy=70 \quad - \ - \quad (3).$$

Adding this to (1), we have,

$$x^2+2xy+y^2=144.$$

Whence, $$x+y=12 \quad - \ - \quad (4).$$

This with (3) will complete the solution; or, subtracting (3) from (1), we have,

$$x^2-2xy+y^2=4.$$

Whence, $$x-y=2.$$

This with (4) gives,

$$x=7,\ y=5.$$

6. Solve $x-y+\sqrt{x-y}=6$

$x+y=8.$

Follow the same general course as in the second example.

Ans. $x=6,\ y=2.$

7. Solve $x^3+y^3=9$ - - - (1),

$x^2y+xy^2=6$ - - (2).

Multiplying (2) by 3 and adding to (1), we have,

$$x^3+3x^2y+3xy^2+y^3=27.$$

Taking the cube root of this,

$$x+y=3 \quad - \ - \quad (3).$$

We may write (2) thus,

$$xy(x+y)=6.$$

This with (3) gives,

$$3xy=6 \quad - \ - \quad (4).$$

From (3) and (4) we have

$$x=1,\ y=2.$$

8. Solve $8x^3+128y^3=3520$

$2x+4y=16.$ *Ans.* $x=2,\ y=3.$

9. Solve $x+y=10$
$\sqrt{xy}=4.$ *Ans.* $x=8,\ y=2.$

10. Solve $\frac{x^2}{y^2}-1=3$
$\frac{x}{y}+1=3.$ *Ans.* $x=6,\ y=3.$

11. Solve $x^2+2xy+y^2=64$
$x^2-y^2=-16.$ *Ans.* $x=3,\ y=5.$

12. Solve $x^2-3x+xy=14+3y$
$x^2+xy+5x=70-5y.$ *Ans.* $x=5,\ y=2.$

13. Solve $3xy+4x-4y^2+10=2y$
$15x^2-6x+4y+3=10xy.$ *Ans.* $x=1,\ y=2.$

14. Solve $x+\sqrt{xy}+y=14$
$x^2+xy+y^2=84.$ *Ans.* $x=2,\ y=8.$

96. Radical Equations.

We shall now give some further examples of the manner of solving equations containing radical quantities. No invariable method can be pointed out.

If there is but one radical in the equation, we may make it stand alone in one member, and then by raising both members to a power equal to the index of the radical, it will disappear. If there are two radicals, by placing them in different members and raising to two successive powers, they may be made to disappear; but generally the resulting equation is of too high a degree to be easily managed.

When there are fractional quantities entering the equation, it may often be greatly simplified by suppressing common factors.

1. Solve $5-\sqrt{x+1}=x.$

Transposing,
$$-\sqrt{x+1}=x-5.$$
Squaring,
$$x+1=x^2-10x+25.$$
Solving,
$$x=8,\ x=3.$$

2. Solve $\frac{x-1}{\sqrt{x}-1}-1=\sqrt{2x-4}.$

Dividing the numerator of the fraction by the denominator,

$$\sqrt{x}+1-1=\sqrt{2x-4}$$

$$\sqrt{x}=\sqrt{2x-4}.$$

Squaring,

$$x=2x-4$$

$$x=4.$$

3. Solve $a+\sqrt{x-b}=\sqrt{x+a^2}$.

Squaring, $$a^2+2a\sqrt{x-b}+x-b=x+a^2.$$

Simplifying, $$2a\sqrt{x-b}=b.$$

Squaring again, $$4a^2(x-b)=b^2.$$

Whence, $$x=\frac{b^2+4a^2b}{4a^2}.$$

4. Solve $\sqrt{\frac{a}{x}}-\frac{\sqrt{x+a}}{b}=\frac{\sqrt{a+x}}{b}$.

Squaring, $$\frac{a}{x}-\frac{2\sqrt{ax+a^2}}{b\sqrt{x}}+\frac{x+a}{b^2}=\frac{a+x}{b^2}.$$

Simplifying, $$\frac{a}{x}-\frac{2\sqrt{ax+a^2}}{b\sqrt{x}}=0.$$

Transposing and squaring,

$$\frac{a^2}{x^2}=\frac{4ax+4a^2}{b^2x}.$$

Clearing of fractions, $a^2b^2=4ax^2+4a^2x$;

whence, $$x^2+ax=\frac{ab^2}{4}.$$

$$x=-\frac{a}{2}\pm\sqrt{\frac{ab^2}{4}+\frac{a^2}{4}},$$

$$x=\frac{-a\pm\sqrt{ab^2+a^2}}{2}.$$

5. Solve $\frac{x-a}{\sqrt{x}-\sqrt{a}}+\frac{\sqrt{x}+\sqrt{a}}{x-a}=\sqrt{x}$.

Getting rid of common factors,

$$\sqrt{x}+\sqrt{a}+\frac{1}{\sqrt{x}-\sqrt{a}}=\sqrt{x}.$$

Clearing of fractions,

$$x-a+1=x-\sqrt{ax}.$$

Simplifying and transposing,

$$\sqrt{ax}=a-1.$$

Squaring and dividing by a, $x=\dfrac{(a-1)^2}{a}$.

6. Solve $\sqrt{x}-\sqrt{x-5}=1$. *Ans.* $x=9$.

7. Solve $a-b=\sqrt{x-2ab}$. *Ans.* $x=a^2+b^2$.

8. Solve $\dfrac{1}{\sqrt{x}-1}-\dfrac{\sqrt{x}+1}{a}=\sqrt{x}+1$. *Ans.* $x=\dfrac{2a+1}{a+1}$.

9. Solve $\sqrt{7x+1}-2=\sqrt{x+11}$. *Ans.* $x=5$.

10. Solve $\dfrac{\sqrt{x}+3}{x-9}=\sqrt{x}-3$. *Ans.* $x=4$.

11. Solve $\sqrt[3]{\dfrac{x+b}{c}}=a$. *Ans.* $x=a^3c-b$.

12. Solve $\sqrt{1+\sqrt{1+x}}=2$. *Ans.* $x=8$.

13. Solve $\sqrt[m]{a+\sqrt[n]{x}}=c$. *Ans.* $x=(c^m-a)^n$.

14. Solve $x^{\frac{1}{2}}+(x-9)^{\frac{1}{2}}=\dfrac{9}{(x-9)^{\frac{1}{2}}}$. *Ans.* $x=12$.

15. Solve $\sqrt{16-\sqrt{x^2+45}}=\dfrac{6}{x}$. *Ans.* $x=2$.

16. Solve $\dfrac{a}{x}+\dfrac{(a^2-x^2)^{\frac{1}{2}}}{x}=\dfrac{x}{b}$. *Ans.* $x=(2ab-b^2)^{\frac{1}{2}}$.

97. Inequalities.

We shall now give a few principles with regard to Inequalities.

Two inequalities are said to subsist in the same sense when the greater quantity is in the same member in both; thus, $5>2$ and $7>3$ are in the same sense. When the greater quantity is in the first member of one, and in the second of the other, they subsist in a contrary sense; thus, $5>2$ and $3<7$, are in a contrary sense.

1. *We may add the same quantity to both members of an inequality, or subtract the same quantity from both members, without changing the sense.*

For example, add 5 to both members, and then subtract it from both members of the following:

$$7>3.$$

We have,

$$12>8, \text{ and } 2>-2.$$

If we subtract 10, we have,

$$-3>-7.$$

This last result is true, considered in an algebraic sense.

This principle enables us to transpose terms from one member of an inequality to the other, by changing the signs, as in an equation.

2. *If both members of an inequality be multiplied or divided by the same positive quantity, the sense will not be changed; but if the multiplier be negative, the sense will be changed.*

For example, multiply both members of $7>5$ by 5. We have,

$$35>25.$$

Multiplying by -5, we have,

$$-35<-25.$$

If we multiply by -1, we change the sense and the signs at the same time; hence, we may change the signs of an inequality if we at the same time reverse the sense.

3. *When both members of an inequality are positive, we may raise to any power; or we may extract any root, provided we use only the positive roots.*

For example, raise both members of $7>5$ to the second power. We have,

$$49>25.$$

Extracting the square root of this last inequality, we have,

$$7>5;$$

but we could not take the negative roots and say

$$-7>-5.$$

These principles will enable us to transform an inequality so as to make any quantity stand alone in one member, greater or less than a resulting quantity; thus, let us have,

$$3x-9>21.$$

Transposing and dividing, we have,

$$x>10.$$

Examples.

1. $2x+5>12$. *Ans.* $x>\frac{7}{2}$.

2. $\frac{3x}{2}-1<x+4$. *Ans.* $x<10$.

3. $\frac{x}{5}-\frac{x}{2}<10-6x$. *Ans.* $x<\frac{100}{57}$.

4. $\frac{x}{a}+b>c-\frac{dx}{a}$. *Ans.* $x>\frac{ac-ab}{d+1}$.

98. Problems.

A problem is a question proposed for solution.

The solution consists,

1. In translating the given conditions into algebraic language, and thus deriving one or more equations, involving the given and the required elements; and

2. In the solution of the resulting equation or equations.

Represent the required elements by x, y, etc., and then use these quantities as though their values were known.

Be careful in forming an equation to put like quantities equal to each other; that is, men=men, pounds=pounds, etc. Be careful, also, that the quantities are expressed in a common unit.

What two numbers are those whose sum is a, and whose difference is b?

Here, although two numbers are required, it is not necessary to use more than one unknown quantity; thus,

Let $x=$ the greater.
Then $a-x=$ the less.

From the condition, we shall have

$$x-(a-x)=b.$$

$$2x=a+b.$$

$$x=\frac{a+b}{2}, \text{ the greater;}$$

$$a-\frac{a+b}{2}=\frac{a-b}{2}, \text{ the less.}$$

We may also use two unknown quantities in the solution of this problem; thus,

Let $x =$ the greater, and

$y =$ the less.

Then $x+y=a$,

and $x-y=b$;

whence $x=\frac{a+b}{2}=\frac{a}{2}+\frac{b}{2}$;

$y=\frac{a-b}{2}=\frac{a}{2}-\frac{b}{2}$.

These results may be translated into English thus:

The greater of two quantities is equal to the half sum, plus the half difference.

The lesser of two quantities is equal to the half sum, minus the half difference.

2. What number is that from which if 5 be subtracted, two-thirds of the remainder will be 40?

Let $x=$the number.

Then, $x-5=$the remainder.

From the conditions, $\frac{2}{3}(x-5)=40$.

$x = 65$.

3. A horse said to a mule: If I give you one of my sacks we shall have an equal number; if I take one of yours, I shall have double the number you have left. How many had each?

Let $x=$ the number the horse had.

$y=$ the number the mule had.

Then, $x-1=y+1$,

and $x+1=2(y-1)$,

whence $x=7$, $y=5$.

4. Divide 64 into two parts which shall be to each other as 3 to 5.

Let $3x=$ one part,

Then $5x=$ the other.

But $3x+5x=64$.

Whence, $x=8$.

$3x=24$ one part,

$5x=40$ the other.

5. Divide a into two parts which shall be to each other as b to c.

Ans. $\frac{ab}{b+c}$, $\frac{ac}{b+c}$.

6. A man left \$2,400 to be divided between two sons and a servant. The sons' parts were to be to each other as 3 to 2, and the servant was to have half as much as the son who received the smaller sum. How much had each? *Ans.* \$1,200, \$800, \$400.

7. A person upon being asked his age, replied: that $\frac{3}{4}$ of his age multiplied by $\frac{1}{12}$ of his age would give a product equal to his age. How old was he? *Ans.* 16 years.

8. A and B have the same income; A contracts an annual debt of $\frac{1}{7}$ of his income; B lives upon $\frac{4}{5}$ of his; at the end of 10 years, B lends A money enough to pay off his debts, and has 160*l* left. What was their income? *Ans.* 280*l*.

9. A man, fifteen years after his marriage, was asked the age of himself and of his wife at their marriage. He replied that he was then twice as old as his wife, but that now he was only once and a half as old. What were their ages? *Ans.* 30 and 15.

10. A person passed $\frac{1}{6}$ of his age in childhood, $\frac{1}{12}$ in youth, $\frac{1}{7}$ and 5 years besides in matrimony, at the end of which time he had a son, who died 4 years before his father, having reached half his father's age. What was the father's age? *Ans.* 84.

11. A privateer running at the rate of 10 miles an hour discovers a ship-of-war 18 miles off pursuing at the rate of 8 miles an hour. How many miles will the privateer make before the ship overhauls her?

Let $x=$ the distance the privateer will make before she is overhauled.
Then, $x+18=$ the ship's corresponding distance.

$\frac{x}{10}$ will be the number of hours required by the privateer to reach the point of union. $\frac{x+18}{8}$ will be the time required by the ship.

Since these times must be equal, we have,

$$\frac{x}{10}=\frac{x+18}{8}, \text{ whence,}$$

$$x=-90.$$

It will be observed that our result is negative. We must, then, reckon backward 90 miles to find the place at which the vessels *would have been* together.

This results from the fact that positive and negative quantities do not differ essentially, but merely in their positions with respect to the origin of reckoning. It is thus that the algebra takes no note of the word "pursuing," except in its essential sense of traveling on the same line with the object with which this relation obtains. So in general, the algebra always interprets words which have relative meanings in their broadest sense. It thus makes no distinction between "will be" and "has been," "up" and "down," "to the right" and "to the left," etc.

A negative result shows that there has been some error made in the use of words in the enunciation. It is to be interpreted in a directly contrary sense from a positive result.

12. Find two numbers such that the first added to three times the second will give 11; and three times the first less twice the second will give zero. *Ans.* 2 and 3.

13. Find three numbers such that three times the first increased by twice the second will give 16; and twice the third less the first will give 6; and five times the third divided by twice the second will give 2. *Ans.* 2, 5, and 4.

14. There are three numerals expressed by single figures which added together give 8; written in a certain order they give five times the number expressed by the second and third written together; if the first and third be written together, they express $\frac{5}{4}$ of the number formed by writing the first and second together. What are the numbers? *Ans.* 1, 2 and 5.

15. What three numbers are those, which if three times the first be added to twice the second and this sum divided by the third, will give 4; and if three times the second be added to five times the third and the sum be divided by the second, will give 7; and if four times the sum of all the numbers be divided by 2, will give 22?

Ans. 2, 5, and 4.

16. A man upon being asked his own and his wife's ages, replied that his age divided by his wife's would give one-fifteenth of her age; but that if 30 had been first subtracted from his own age, the result would have been one-thirtieth of his wife's age. How old was each?

Ans. 60 and 30.

17. The length of a rectangular field exceeds its breadth by 10 chains, and it has 2000 square chains in it. What is its length and breadth? *Ans.* 40 and 50.

18. What two numbers are those which if the first be taken from

the second and the difference multiplied by the first, the result will be four times the square of the first; and if this difference be squared, the result will be 64? *Ans.* 2 and 10.

19. A merchant has two sorts of tea, one worth 75 cents a pound and the other one dollar. He wants to make a mixture of 25 pounds which he can sell at 90 cents. How many pounds of each must he take? *Ans.* 10 of the 1st and 15 of the 2d.

20. A hare is 50 leaps before a greyhound, and makes 4 leaps to the greyhound's three; but two of the hound's leaps are equal to three of the hare's. How many leaps must the greyhound make to catch the hare?

In such a case as this, we must first fix upon a unit of distance, which here may be either one leap of the dog or one of the hare. Let us take the length of the hare's leap as the unit; then, the length of the dog's leap will be $\frac{3}{2}$ times that of the hare's.

Now, if x be the number of leaps the hound must make, the hare will in the same time make $\frac{4}{3}x$ leaps. This number of leaps made by the hare, added to the number of leaps it is in advance, will be the distance, in our assumed unit, from where the dog is to the place where the hare is caught. But the dog will pass over a distance equal to $\frac{3}{2}x$; that is, the number of leaps multiplied by the length of one of them; hence,

$$\tfrac{4}{3}x+50=\tfrac{3}{2}x. \quad \therefore$$
$$x=300. \quad \textit{Ans.}$$

21. Two trains are traveling towards each other at the rate of 20 and 33 miles per hour, respectively; they are 265 miles apart. How long before they will meet? *Ans.* 5 hours.

22. If the trains had been traveling at the rate of a and b miles per hour and had been c miles apart, how long would they take to come together? *Ans.* $x=\dfrac{c}{a+b}$.

From this formula, tell at once what the time would have been had the rates been 2 and 3, and the distance 10. Also, if the rates had been $\frac{1}{2}$ and $\frac{1}{3}$, the distance $\frac{1}{7}$. Also, if the rates had been 5 and -7, the distance 6.

23. Two travelers set out at the same time to meet each other, being 154 miles apart. If one travels at the rate of 3 miles in 2 hours, and the other at the rate of 5 miles in 4 hours, how far shall each travel before they meet, and after what time?

Ans. Distances, 84 and 70. Time, 56 hours.

Let letters be used for the numerals, and the results be applied to particular cases, as above. So with any of the preceding or following problems.

24. A can do a piece of work in 5 days, and B can do the same in 7; working together, how long will they be at it?

Ans. $2\frac{11}{12}$ days.

25. The half of A's fortune less B's is equal to \$3,000; B's fortune less one-fifth A's gives zero. What sum had each?

Ans. A, \$10,000, B, \$2,000.

26. What two numbers are those which cubed and added together give 35, and which if the first be squared and multiplied by the second, and this result be added to the product of the second squared by the first, will give 30? *Ans.* 2 and 3.

SECTION VIII.

SYMBOLS 0 AND ∞.—DISCUSSIONS.

99. The entire absence of value is represented by the symbol 0, called *zero*. A man who is altogether destitute of money, has 0 dollars. This is the ordinary meaning of *zero*; but in algebra it has a somewhat more extended signification.

A man may not only be destitute of money, but he may have less than none at all, in an algebraic sense, as we have seen in Art. 26; he may be in debt. In such a case, *zero* is the point of division between his assets and liabilities.

Again, if we agree to reckon distances upon a right line or scale from a particular point upon it, such point, having no distance from the origin of distances, is the zero point. We have a familiar example of this in the thermometer; the position of the zero point upon the scale being a mere matter of agreement.

Thus, *zero* in algebra is but the point of separation between all positive quantities on the one hand, and all negative quantities on the other. If a positive quantity be continually diminished, it will approach zero as a limit; but as we cannot conceive of a quantity so small that there cannot be a smaller, we could never find the smallest possible quantity. But that which continually approaches a limit may be said to have the limit itself for its extreme value; and so

0 *is often said to be a quantity smaller than any assignable quantity.*

The symbol ∞, called infinity, represents a quantity than which a greater is impossible. It is the limit of all augmentation. We may approach it to any possible degree of approximation, but can never reach it.

100. Combinations of 0 and ∞.

Any product which has 0 in it as a factor, must itself be *zero;* thus,

$$a \times 0 = 0.$$

For if zero be taken any number of times, it will still be *zero;* or if any quantity be taken *zero* times, we shall have zero as well.

Zero divided by any finite quantity is equal to zero; thus,

$$\frac{0}{a} = 0.$$

For, since the product of the divisor and quotient must produce the dividend, we must have 0 for the quotient to give with a, the dividend 0; or, more briefly, zero times the fractional unit, gives zero.

A finite quantity divided by zero, gives infinity; thus,

$$\frac{a}{0} = \infty.$$

For, as we diminish the divisor, we increase the quotient. When we have made the divisor the least possible, the quotient must be the greatest possible, that is, *infinite.*

Zero divided by zero, gives any quantity whatever, or, as it is said, is *Indeterminate;* thus,

$$\frac{0}{0} = a.$$

For, any quantity, a, the quotient, multiplied by the divisor 0, gives 0 the dividend.

Any quantity divided by infinity gives zero; thus,

$$\frac{a}{\infty} = 0.$$

For, as the divisor is increased, the quotient is diminished. When the divisor is the greatest possible, the quotient must be the least possible, or *zero.*

Infinity divided by a finite quantity gives infinity; thus,

$$\frac{\infty}{a} = \infty.$$

For, when the dividend is the greatest possible, the divisor being constant, the quotient must also be the greatest possible, or infinite.

Thus we may say that,

1. *Zero multiplied by any quantity is zero;*
2. *Zero divided by a finite quantity is zero;*
3. *A finite quantity divided by zero is infinite;*
4. *Zero divided by zero is any quantity;*
5. *A finite quantity divided by infinity is zero;*
6. *Infinity divided by a finite quantity is infinite.*

101. Vanishing Fractions.

1. Sometimes an algebraic fraction reduces to the form $\frac{0}{0}$, under a certain hypothesis made upon the quantities which enter it, when 0 is not the true value of the fraction; thus,

$$y=\frac{(a^3-x^3)^2}{a^2-x^2}$$

when $a=x$, becomes,

$$y=\frac{0}{0}.$$

This is not the true value of y; for, before making the hypothesis, resolve the numerator and denominator into their factors; thus,

$$y=\frac{(a-x)(a^2+ax+x^2)}{(a+x)(a-x)},$$

and cancel the common factor, $x-a$; thus,

$$y=\frac{a^2+ax+x^2}{a+x}.$$

Now make $a=x$, and we have,

$$y=\frac{3a^2}{2a}=\frac{3a}{2}, \text{ the true value of } y.$$

2. Again,

$$y=\left(\frac{a^2-x^2}{(a-x)^2}\right)_{x=a}{}^*=\frac{0}{0}.$$

But, $$y=\frac{(a+x)(a-x)}{(a-x)(a-x)}=\left(\frac{a+x}{a-x}\right)_{x=a}=\frac{2a}{0}=\infty.$$

3. Again,

$$y=\left(\frac{(a-x)^2}{a^3-x^3}\right)_{x=a}=\frac{0}{0}.$$

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* This notation shows that a is to be made equal to x in the expression; read, *when* x *is equal to* a.

But,

$$y=\frac{(a-x)(a-x)}{(a-x)(a^2+ax+x^2)}=\left(\frac{a-x}{a^2+ax+x^2}\right)_{x=a}=\frac{0}{3a^2}=0.$$

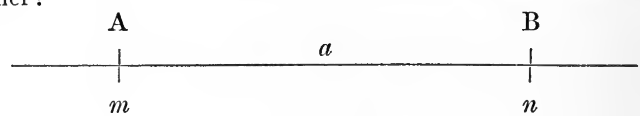
Such expressions are called *Vanishing Fractions.* They appear to be equal to $\frac{0}{0}$ for a particular hypothesis, but really are not so. They reduce to this form from the presence of a concealed common factor which becomes zero under the particular hypothesis. When such factor is stricken out, the true value results under the hypothesis. As we have seen, the true value may be either $\frac{a}{b}$, $\frac{a}{0}$, or $\frac{0}{a}$.

102. The Problem of the Couriers.

The *discussion* of an expression consists in making every possible supposition upon the arbitrary quantities which enter it, and interpreting the results.

The following problem gives rise to some interesting results.

Two couriers traveling on the same straight line, one at the rate of m miles an hour, and the other at n miles an hour, are separated by a distance of a miles at 12 o'clock M. When will they be together?



Let one of the couriers be at A and the other at B. The distance from A to B will be a.

Now, let the couriers be moving to the right, and let us reckon all distances from A as the origin of distances,—those to the right of A being positive, and those to the left negative.

Let the courier at A be traveling at m miles an hour, and the one at B at the rate of n miles an hour. Using two unknown quantities, let t be the number of hours, counting from 12 M. until the union takes place, and x the number of miles to be traveled by the courier at B to reach the place of meeting. Then we shall have,

$$nt=x \quad - \quad - \quad - \quad (1),$$

and

$$mt=x+a \quad - \quad - \quad (2).$$

Combining these equations and eliminating x, we have,

$$mt-nt=a.$$

$$t=\frac{a}{m-n}.$$

This value of t in (1) gives,

$$x=\frac{na}{m-n}$$

for the distance traveled by the foremost courier. Now, let us take the root, $t=\dfrac{a}{m-n}$, and make every possible hypothesis upon a, m, and n, and ascertain what t will denote in each case.

I. *First let* a *be a positive quantity, and* m>n.

In this case the denominator of the fraction which expresses the value of t, will be positive, and since a is also positive, t must be essentially positive.

We shall have to add t hours, therefore, to 12 M. in order to find the time at which the couriers will be together.

This is what common sense would tell us, since we simply have the case of one courier pursuing the other at a more rapid pace, and so must be constantly gaining on him and at last must overtake him.

II. *Let* a *be positive, and* m<n.

In this case the denominator of the fraction is negative and the numerator positive; t is, therefore, negative.

We must, hence, subtract t hours from the origin of time, 12 o'clock; that is, reckon backwards to find the time at which the couriers *were together*.*

This we can readily understand, also; for since the courier in advance is, under the hypothesis, traveling more rapidly than the one in the rear, a moment before 12 o'clock, there was less distance between them, and less the moment before that, and so there must have been a time when they were together.

III. *Let* a *be positive, and* m=n.

In this case the denominator of the fraction will be 0: hence t is ∞.

This is plainly as it should be, since with a certain distance between them, and traveling at the same rate, they cannot be together in any finite time. The result ∞ is here equivalent to *never*.

IV. *Let* a *be negative, and* m>n.

In this case, the denominator of the fraction is positive, but the numerator negative. t is, therefore, negative. We must reckon backwards to find the time of conjunction.

* See remarks under problem 11, page 110.

a being negative, places the courier who is traveling at the rate of n miles an hour, in rear of the other; and since he is moving more slowly, they must have been already together in the past.

V. *Let* a *be negative, and* $m<n$.

t will be positive, and the meeting has yet to take place.

VI. *Let* a *be negative, and* $m=n$.

t is infinite, and may be either positive or negative. The couriers never have been, and never will be together; or we may say they were together, and that they will be together an infinite time either in the past or future.

VII. *Let* $a=0$, *and* $m>n$ *or* $<n$.

t is zero; and the couriers were together at 12 o'clock. They manifestly never could have been together before, and can never be again.

VIII. *Let* $a=0$, *and* $m=n$.

t becomes $\frac{0}{0}$.

That is, any time may be added to or subtracted from the epoch, 12 o'clock.

Since the couriers are together at 12 o'clock, and are moving at the same rate, they have always been, and will always be together.

IX. *Let* $a=0$, $m=0$, *and* $n=0$.

Under this hypothesis, we have again $t=\frac{0}{0}$. Here the couriers are together and are not moving at all. Of course they always have been and always will be together.

X. We may now fix the value of any three of the four quantities, t, a, m, or n, at pleasure, and thus determine the remaining one. Let the instructor make such hypotheses and require the student to determine and interpret the results.

We have not yet entirely exhausted the possible hypotheses upon the quantities which enter the expression under consideration. We may make m and n negative in succession; in which case the couriers would be traveling in opposite directions; or we may make them both negative at the same time, in which case they would be traveling to the left. Let the instructor give such exercises.

103. The Problem of the Lights.

The discussion of the *Problem of the Lights*, as it is called, also gives rise to many interesting and instructive results.

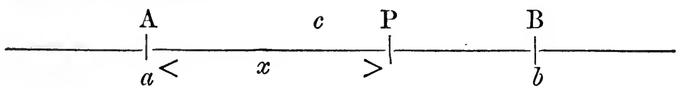
The problem is this:

Given, two lights, placed anywhere upon a straight line, to determine the point or points which will be equally illuminated by each.

We must first know, obviously, the law which governs the illuminating power of light; and, upon this point, physics teaches us, that,

The intensity of a light at any distance is equal to its intensity at the distance 1, *divided by the square of the given distance;* that is, if a light has an intensity, say 10, at the distance of one yard from it, at two yards its illuminating power will be $\frac{10}{2^2}$; at three yards, $\frac{10}{3^2}$; at x yards, $\frac{10}{x^2}$.

Then let the lights in question be placed upon the line A B, one at A, and the other at B.



Let the distance between them be c, and let the intensities of the lights at the unit of distance be, respectively, a and b. Suppose the point P to be one point equally illuminated, and let its distance from A, the origin of distances, be x; its distance from B will be $c-x$. Let distances to the right of A be positive; those to the left will be negative.

Now, the illuminating power of the light A, for the point P, will be $\frac{a}{x^2}$; and that of the light B for the same point, will be $\frac{b}{(c-x)^2}$. Since the quantity of light from each of the lights for this point is to be the same, we must have

$$\frac{a}{x^2}=\frac{b}{(c-x)^2}.$$

Solving this equation, we have

$$x=\frac{c\sqrt{a}}{\sqrt{a}+\sqrt{b}}, \text{ and}$$

$$x=\frac{c\sqrt{a}}{\sqrt{a}-\sqrt{b}}.$$

Since we find two roots of the equation, there must be two points of equal illumination, so long, at least, as the roots are of different values.

Now, to discuss these two expressions,

I. *Let* c *be positive, and* a $>$b.

The first root will manifestly be positive; the point corresponding to this value of x will, therefore, be found somewhere to the right of A. Its distance from A will be c times the fraction $\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}$. But since a is greater than b in this case, the denominator is less than twice the numerator, and, hence, this fraction is greater than $\frac{1}{2}$. c times the fraction must, therefore, be greater than $\frac{1}{2}c$. x is thus greater than $\frac{c}{2}$, or the point P must be farther from A than it is from B.

This is what common sense would teach; for, the stronger light being at A, the point equally illuminated must be nearer the lesser light.

The second root under this hypothesis is also positive, and since in the fraction, $\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}$, the denominator is less than the numerator, the fraction will be greater than unity, and consequently c times it will be greater than c. x is thus greater than c; that is, this second point lies beyond B, the feebler light.

This must be so, since, as we move to the right of the lesser light, the difference between the quantity of light from the two sources becomes less and less. It will, thus, at last be zero; that is, we shall reach a point which will receive the same quantity of light from each.

II. *Let* c *be positive and* a$<$b.

Under these hypotheses, the first root will still be positive, but it will be less in value than c; that is, the corresponding point will lie nearer to A, now the feebler light.

The second value of x will be negative; that is, the second point will lie to the left of A.

In this case we have but changed the places of the lights, and of course the circumstances are just reversed from those in the previous case.

III. *Let* c *be positive and* a$=$b.

In this case the first root becomes, $x=\frac{c\sqrt{a}}{2\sqrt{a}}=\frac{c}{2}$; that is, the first point is midway between the lights. Since the lights are equal, under the hypothesis, this is manifestly true.

The second root becomes,

$$x=\frac{c\sqrt{a}}{0}=\infty.$$

The farther we recede from the lights, the less will be the difference in the quantity of light at any point; but this difference will not become zero, or, in other words, the quantity of light will not be entirely equal until we have reached an infinite distance, which, of course, can never be.

IV. *Let* c *be negative and* a > *or* < b, *or* a=b.

The effect of making c negative, is to place the light whose intensity is b on the left of the other, the light having the intensity a being still the origin of distances. The separate discussion of the several relations of a and b to each other in this case would give the reverse of the results already considered.

V. *Let* c=0 *and* a> *or* <b.

Both roots in this case become 0, and thus the origin is equally illuminated.

[The supposition that c is 0, places the two lights together at the origin. The lights, however, are of different powers. It is thus laid down in several late works on the subject, that this point cannot be equally illuminated by the two lights, and consequently that the analysis fails for this case. Several attempts have been made to explain the anomaly; but in reality it should seem that there is no failure and no real difficulty.

Let it be remembered that our equation is deduced under a postulate borrowed from physics with regard to the effect of light at different distances. The algebra accepts that law as absolutely true, and its results must be interpreted strictly under that hypothesis.

Let us then approach, say, the light whose intensity is a at the unit's distance. When we have reached the distance $\frac{1}{2}$, its intensity is, by the law, $\frac{a}{(\frac{1}{2})^2}=4a$; at the distance $\frac{1}{100}$, it is $10{,}000a$; at 0 it is ∞. The light whose intensity is b, or has any other intensity at the unit's distance, is also ∞ at the point zero. Since, then, under our assumed law, both lights are infinite at the point 0, that point is equally illuminated, and the roots are truly zero. The results of the analysis may be further vindicated by using mathematical lines, and thus removing the question entirely beyond the laws of physical science. See note at the end of the book.]

VI. *Let* c=0 *and* a=b.

The first root becomes 0, and the second $\frac{0}{0}$; that is, one of the points is at the origin, and the other anywhere we please.

This is easily understood.

VII. *Let* a *and* b *be negative in succession, or together.*

In this case both roots are imaginary.

This is as it ought to be, since under our assumed law of physics the absence of light, or total darkness, cannot be reached until the distance from the source becomes infinite. There is thus an infinite separation between absolute light on the one hand, and no light at all on the other. Less than no light is, therefore, impossible.

VIII. *Let* a=0, b=0, c=0.

Both roots become $\frac{0}{0}$, and thus all points are equally illuminated, and this must be true since no point would have any light at all.

104. General Properties of Equations of the Second Degree.

Equations of the second degree containing but one unknown quantity possess some important properties which we shall now proceed to investigate.

Resuming the general equation,

$$x^2+2px=q \quad - - - - - \quad (1),$$

let us complete the square and transpose all the terms into the first member. We shall have,

$$x^2+2px+p^2-(q+p^2)=0.$$

or,

$$(x+p)^2-(q+p^2)=0.$$

Regarding this as the difference of two squares, we may write,

$$(x+p-\sqrt{q+p^2})(x+p+\sqrt{q+p^2})=0 \quad - - - - - \quad (2).$$

This equation can be true only upon the supposition that one of the factors composing the first member is equal to zero.

We may, then, have either,

$$x+p-\sqrt{q+p^2}=0, \quad \text{or,}$$

$$x+p+\sqrt{q+p^2}=0.$$

From these we get,

$$x=-p+\sqrt{q+p^2}: \text{ the first root, and}$$
$$x=-p-\sqrt{q+p^2}: \text{ the second root.}$$

As there are no other possible values of x which will satisfy this equation, we may say that,

1*st. Every equation of the second degree has two roots and only two.*

By inspection of equation (2) we see that,

2*d. The first member of every equation of the second degree, when the second member is* 0, *is composed of two factors, having the unknown quantity for the first term in each, and the respective roots with their signs changed for the other terms.*

If the roots are given, the equation can be constructed at once; thus, for example, let the roots of a certain equation be $x=a$ and $x=-b$. The equation will be,

$$(x-a)(x+b)=0, \text{ or}$$
$$x^2+bx-ax-ab=0.$$

EXAMPLES.

The roots being 4 and -5; what is the equation?

Ans. $x^2-x-20=0$.

The roots being ab and $-c$: a and a^2: -7 and 8: -6 and -5: $\frac{a}{b}$ and $-\frac{c}{d}$: -1 and $+1$: $\sqrt{-1}$ and $-\sqrt{-1}$: what are the equations in these several cases?

105. The Sum of the Roots.

Let us now add together the two roots,

$$x' =-p+\sqrt{q+p^2}$$
$$x''=-p-\sqrt{q+p^2}$$

using x' and x'' to distinguish them. We get,

$$x'+x''=-2p.$$

Hence we may say, that,

3*d. The sum of the two roots is equal to the co-efficient of the first power of the unknown quantity with its sign changed.*

106. The Product of the Roots.

Now, multiplying the roots together,

$$x' = -p+\sqrt{q+p^2}$$

$$x''= -p-\sqrt{q+p^2}\text{; we have,}$$

$$x'x''= -q.$$

Hence, we may say that,

4th. The product of the two roots is equal to the second member with its sign changed.

107. The Greatest Numerical Value of q when Negative.

Since the product of the two roots is always equal to q (the second member), with a contrary sign, if q is negative, the roots must have like signs; and when added together, the algebraic sum will be their *numerical* sum. Now, this sum is, as we have seen, equal to $2p$.

We thus have the sum of two quantities given: and now let us find how to divide this sum into two parts, so that their product shall be the greatest possible.

Let d be the difference between the two parts, $2p$ being the sum.

Then, the greater will be (Prob. 1, Art. 98),

$$p+\frac{d}{2}.$$

and the less,

$$p-\frac{d}{2};$$

Their product must be equal to q, and we shall have,

$$p^2-\frac{d^2}{4}=q.$$

Now, as d is diminished, q will increase, until when $d=0$, q will be equal to p^2, and will then be the greatest possible: that is to say, when the roots are equal, their product will be the greatest possible. q, therefore, can never be greater than p^2. This, however, is under the supposition that q is negative.

When q is positive, since the roots multiplied together must then give $-q$, they will have contrary signs, and when added, will give their numerical difference, instead of their numerical sum. There is, then, no limit to the values of q in such a case.

Hence, we may say, that,

5th. *When the second member is negative, it can never be numerically greater than the square of one half the co-efficient of the first power of the unknown quantity.*

108. Discussion of the Four Forms.

Resuming the *Four Forms* already given (Art. 82), viz.:

$$x^2+2px=\ \ q \quad - - - - - \quad (1).$$
$$x^2-2px=\ \ q \quad - - - - - \quad (2).$$
$$x^2+2px=-q \quad - - - - - \quad (3).$$
$$x^2-2px=-q \quad - - - - - \quad (4).$$

Writing the root, respectively, we have,

$$x=-p\pm\sqrt{q+p^2} \quad - - - \quad (1).$$
$$x=+p\pm\sqrt{q+p^2} \quad - - - \quad (2).$$
$$x=-p\pm\sqrt{-q+p^2} \quad - - \quad (3).$$
$$x=+p\pm\sqrt{-q+p^2} \quad - - \quad (4).$$

Since $\sqrt{q+p^2}$ is greater than p, the radical parts of the roots in the first and second forms are greater, numerically, than their entire parts. The radical parts will therefore govern the signs in these two forms; so that in the first and second forms the signs of the roots will be unlike, and the negative root will be numerically the greater in the first, while the positive root will be the greater in the second.

In the third and fourth forms, the roots will be imaginary when q is numerically greater than p^2. This should be the case, since we have proved that it is impossible for q, when negative, to be greater than p^2. When q has such a value, it shows that the equation from which the roots came is impossible.

When q is less than p^2, the radical parts in the last two forms will be less than p, the entire parts; so that when the roots are real in the third and fourth forms, they are both negative in the third, and both positive in the fourth.

If $q=p^2$ in the third and fourth forms, the radical parts of the roots reduce to zero. The roots will then be equal in either form, both being $-p$ in the third, and $+p$ in the fourth form.

When $p=0$, the roots in the first two forms reduce to $\pm\sqrt{q}$, and in the last two $\pm\sqrt{-q}$. Under this supposition, the roots are thus equal with contrary signs in the first and second forms, and are always imaginary in the third and fourth. This ought evidently to be the case, since by making $p=0$ in the forms themselves, they reduce to two sets of incomplete equations; thus,

$$x^2 = q$$
$$x^2 = -q.$$

Making $q=0$, the roots of the first and third forms become,

$$x = -p \pm p.$$

Those of the second and fourth,

$$x = +p \pm p.$$

Or,
$$x = 0, \text{ and } x = -2p$$
$$x = +2p, \text{ and } x = 0.$$

Thus, under this hypothesis, one of the roots in each form becomes zero.

The reason of this readily appears, for if q be made 0 in the forms themselves, we have, for the first and third,

$$x^2 + 2px = 0; \quad - \quad - \quad (a)$$

and for the second and fourth,

$$x^2 - 2px = 0. \quad - \quad - \quad (b)$$

These may be written,

$$x(x+2p) = 0$$
$$x(x-2p) = 0.$$

We may satisfy these, by making either one of the factors zero; thus,

$$x=0 \text{ or } (x+2p)=0$$
$$x=0 \text{ or } (x-2p)=0.$$

Whence the roots are,

$$x=0 \text{ and } x=-2p$$
$$x=0 \text{ and } x=+2p.$$

We might at once divide out x from equations (a) and (b), and thus reduce them to equations of the first degree; and, generally, *whenever we can divide one equation through by the unknown quantity, one of its roots is zero.*

If $p=0$ and $q=0$, the roots are 0.

This supposition reduces the equations themselves to

$$x^2 = 0 \quad \therefore \quad x=0 \text{ and } x=0.$$

Much the same discussion may be had from the forms themselves, by the use of the 3d and 4th general properties established in Articles 105 and 106.

For example, in the first form the roots must have unlike signs, because their product must give $-q$; they are unequal in numerical value, and the negative is the greater, because their algebraic sum must give $-2p$.

Let the student carry on the discussion.

SECTION IX.

ARITHMETICAL PROGRESSION.—RATIO AND PROPORTION.—GEOMETRICAL PROGRESSION.

109. A series is a succession of terms, any one of which may be derived from the preceding term or terms according to a uniform law, called the *law of the series.*

110. Arithmetical Progression.

An Arithmetical Progression is a series in which any term may be derived from the one preceding it by adding a constant quantity, called the *common difference;* thus,

$$1,\ 3,\ 5,\ 7,\ 9\ \ -\ -\ \ \text{etc.},$$

is such a progression, the common difference being 2. When the common difference is *positive,* as in this case, we have an *increasing* progression. When the common difference is *negative,* we shall have a *decreasing* progression; thus,

$$9,\ 7,\ 5,\ 3,\ 1,\ -1,\ -3\ \ -\ -\ \ \text{etc.},$$

is a decreasing progression, in which -2 is the common difference.

111. Formula for the Last Term.

Any term with which we choose to begin the series is called the *first term;* that with which we end it, is called the *last term.* These two are the *extremes.*

Let

$$a,\ b,\ c,\ e,\ f\ \ -\ -\ \ \text{etc.},$$

be an arithmetical progression, in which d is the common difference. Then we must have,

$$b=a+d,\quad c=b+d=a+2d,$$
$$e=c+d=a+3d;\ \text{etc.},$$

and it is evident that any term may be found by adding to the first term as many times the common difference as there are preceding terms.

Then if l represent the last term, and n the number of terms, we must have,

$$l=a+(n-1)d.$$

If d be negative, the formula will be,

$$l=a-(n-1)d.$$

Hence, we may say that,

To find any term of an arithmetical progression, multiply the common difference by the number of preceding terms, and add this product to the first term, if the progression is increasing, or subtract it if it is decreasing.

112. The Sum of Equi-distant Means.

Let us have an increasing progression of a definite number of terms. If t represent the term which has m terms before it, and d the common difference, we shall have,

$$t=a+md \quad - \ - \ - \ - \quad (1).$$

Now, if we reverse the order of terms, we shall have a decreasing progression, with $-d$ for the common difference. If t' represent the term which now has m terms before it, we shall have

$$t'=l-md \quad - \ - \ - \ - \quad (2).$$

Adding (1) and (2), we have

$$t+t'=a+l.$$

Hence, we conclude, that

In any arithmetical progression, the sum of the two terms which are at equal distances from the extremes, is equal to the sum of the extremes themselves.

113. Formula for the Sum.

If s represent the sum of n terms of a progression, we shall have,

$$s=a+b+c+ \quad - \ - \ - \ - \quad j+k+l.$$

Reversing the order, we shall have,

$$s=l+k+j+ \quad - \ - \ - \ - \quad c+b+a.$$

Adding these equations, member to member, we have,

$$2s=(a+l)+(l+k)+ \quad - \quad - \quad - \quad - \quad - \quad (k+b)+(l+a).$$

The terms taken two and two, as shown, give equal sums, from the principle just established; and there will be as many of these partial sums as there are terms in the progression; so that we shall have,

$$2s=(a+l)n; \text{ whence}$$

$$s=\frac{(a+l)n}{2}.$$

Hence, we may say, that,

The sum of a definite number of the terms of an arithmetical progression, is equal to half the sum of the extremes, multiplied by the number of terms.

The two formulas,

$$l=a+(n-1)d \quad - \quad - \quad - \quad - \quad (1),$$

$$s=\frac{(a+l)n}{2} \quad - \quad - \quad - \quad - \quad (2),$$

are sufficient to solve all ordinary questions touching an arithmetical progression. There are altogether five arbitrary quantities,

$$a, \ l, \ d, \ n, \ s,$$

entering these formulas. We may assume any three of them at pleasure, and, regarding the other two as unknown, we may combine the equations, and thus deduce their separate values.

For example, let d, n, and s be given to find a and l.

Substitute the value of l from (1) in (2), we shall have,

$$s=\frac{[a+a+(n-1)d]n}{2}.$$

From this,

$$a=\frac{2s-(n-1)dn}{2n}.$$

This value of a in (1), gives,

$$l=\frac{2s+(n-1)dn}{2n}.$$

Let the student be required to find the formulas for determining any two of the elements when the other three are given by the instructor.

EXAMPLES.

1. In the progression 1, 2, 3, etc., of 14 terms, what is the last term, and what the sum of the terms? *Ans.* $l=14$, $s=105$.

2. In 2, 5, 8 - - - - - of 17 terms, find l and s.
Ans. $l=50$, $s=442$.

3. In 7, $7\frac{1}{4}$, $7\frac{1}{2}$, - - - - of 16 terms, find l and s.
Ans. $l=10\frac{3}{4}$, $s=142$.

4. In $\frac{1}{2}$, $\frac{3}{8}$, $\frac{2}{8}$, - - - - of 20 terms, find l and s.
Ans. $l=-1\frac{7}{8}$, $s=-13\frac{3}{4}$.

5. In 0, $\frac{1}{2}$, 1, - - - - of 11 terms, find l and s.
Ans. $l=5$, $s=27\frac{1}{2}$.

6. In -10, -12, -14, - - - - of 6 terms, find l and s.
Ans. $l=-20$, $s=-90$.

7. Given $a=2$, $n=5$, $l=22$, to find d and s.
Ans. $d=5$, $s=60$.

8. Given $d=2$, $n=12$, $s=96$, to find a and l.
Ans. $a=-3$, $l=19$.

9. One hundred stones being placed on the ground in a straight line, two yards apart, how far will a person travel who shall bring them one by one to a basket, placed at two yards from the first stone? *Ans.* 20,200 yds.

10. A railway train moves two yards the first second, four yards the second second, six yards the third. In how long a time will the train be traveling at the rate of a mile a minute?
Ans. 14.66 sec.

114. Ratio.

Ratio is the relative magnitude of two quantities of the same kind. The measure of this relationship, or, as it is commonly said, the ratio itself, may be always found by dividing one of the quantities by the other; thus, if a and b are the quantities, then $\frac{b}{a}$ or $\frac{a}{b}$, expresses the number of times the one contains the other, and is their ratio, or the measure of their relative magnitudes.

There is some difference in usage as to which quantity shall be made the divisor; thus, the ratio of $3:6$ is about as often written, $\frac{3}{6}$ as $\frac{6}{3}$. The question depends upon which of the two is regarded as the standard; the ratio in this case being $\frac{1}{2}$, if 6 is taken as the measure; or 2, if 3 is so taken. It is perhaps better to make the quantity mentioned first, called the *antecedent*, the divisor, and the second quantity, called the *consequent*, the dividend. At any rate, we shall adopt this method.

115. Proportion.

When two ratios are equal to each other; as,

$$\frac{b}{a}=\frac{d}{c}, \quad - \ - \ - \ - \quad (1)$$

the four quantities are said to be in proportion. They are often written,

$$a : b :: c : d.$$

This and equation (1) express entirely the same truth, and the one may at any time be used for the other.

We may say, then, that,

A proportion is an equality of two ratios.

The ratios are called *couplets;* thus, $a : b$ is the *first couplet;* $c : d$ is the *second couplet.* Of the four quantities in a proportion, the last one is called a *fourth proportional* to the other three. If the second term is used also as the third, such term is called a *mean proportional.* In this case the last term is called a *third proportional.* The first and last terms are called *extremes;* the second and third are called *means.*

116. Let us have the proportion,

$$a : b :: c : d;$$

or, writing it as an equation,

$$\frac{b}{a}=\frac{d}{c};$$

whence,

$$bc=ad. \quad - \ - \quad (1)$$

Hence, we may say that,

The product of the means is equal to the product of the extremes.

The converse of this is equally true; for dividing each member of (1) by ac, we have,

$$\frac{b}{a}=\frac{d}{c}; \text{ or } a : b :: c : d.$$

Whence,

If the product of two quantities is equal to the product of two other quantities, two of them may be made the extremes, and two the means of a proportion.

117. Assume again the proportion,

$$a : b :: c : d; \text{ or}$$

$$\frac{b}{a} = \frac{d}{c}. \quad - \quad - \quad (1)$$

We may multiply both terms of each fraction by any quantity, as m; thus,

$$\frac{mb}{ma} = \frac{md}{mc}; \therefore ma : mb :: mc : md. \quad - \quad - \quad - \quad (2)$$

We may also divide both terms of each fraction by any quantity, as m; thus,

$$\frac{\frac{b}{m}}{\frac{a}{m}} = \frac{\frac{d}{m}}{\frac{c}{m}}; \therefore \frac{a}{m} : \frac{b}{m} :: \frac{c}{m} : \frac{d}{m}. \quad - \quad - \quad - \quad - \quad - \quad - \quad (3)$$

We may extract any root, as the mth, of both members of (1), and shall have,

$$\frac{\sqrt[m]{b}}{\sqrt[m]{a}} = \frac{\sqrt[m]{d}}{\sqrt[m]{c}}, \quad \therefore \sqrt[m]{a} : \sqrt[m]{b} :: \sqrt[m]{c} : \sqrt[m]{d}. \quad - \quad - \quad (4)$$

Or, again, we may raise both members of (1) to the mth power, thus,

$$\frac{b^m}{a^m} = \frac{d^m}{c^m}; \therefore a^m : b^m :: c^m : d^m. \quad - \quad - \quad - \quad - \quad - \quad (5)$$

Hence, from (2), (3), (4), and (5), it follows that,

1. *We may multiply all the terms of a proportion by the same quantity.*
2. *We may divide all the terms of a proportion by the same quantity.*

Remark.—It is plain that the multiplier or divisor may be different for each couplet.

3. *We may extract the same root of every term of a proportion.*
4. *We may raise every term to the same power.*

118. Dividing unity by each member of the equation

$$\frac{b}{a}=\frac{d}{c}; \text{ we have,}$$

$$\frac{a}{b}=\frac{c}{d}-; \quad \therefore b : a :: d : c.$$

Whence,

Consequents may be made antecedents and antecedents consequents. The quantities are then said to be in proportion by *inversion.*

Multiplying both members of

$$\frac{b}{a}=\frac{d}{c}, \text{ by } \frac{c}{b}; \text{ we have,}$$

$$\frac{c}{a}=\frac{d}{b}, \quad \therefore a : c :: b : d.$$

Whence,

The antecedents may be made one couplet, and the consequents another. The quantities are then said to be in proportion by *alternation.*

119. Adding unity to both members of

$$\frac{b}{a}=\frac{d}{c},$$

and then subtracting unity from both, we have,

$$\frac{b}{a}+1=\frac{d}{c}+1; \quad \frac{b}{a}-1=\frac{d}{c}-1.$$

Whence, $$\frac{b+a}{a}=\frac{d+c}{c}; \quad \frac{b-a}{a}=\frac{d-c}{c}.$$

From which we may write,

$$a : b+a :: c : d+c; \quad a : b-c :: c : d-c.$$

Hence, we may say that,

The first antecedent may be added to its consequent, provided the second antecedent is added to its consequent. The quantities are then said to be in proportion by *composition.* In the same way the consequents may be subtracted. The quantities are then said to be in proportion by *division.*

Let us have two proportions with a couplet the same in each; thus,

$$\frac{b}{a}=\frac{d}{c}; \quad \frac{b}{a}=\frac{g}{f}.$$

Then, $$\frac{d}{c}=\frac{g}{f}. \quad \therefore$$

$$c : d :: f : g.$$

That is,

If the first couplets are the same in two proportions, the other two couplets form a proportion.

120. From $\frac{b}{a}=\frac{d}{c}$ we may write, $\frac{mb}{ma}=\frac{nd}{nc}$. Now, making $m=1\pm\frac{p}{q}$, and $n=1\pm\frac{r}{s}$, in this equation we shall have, after a slight transformation,

$$\frac{b\pm\frac{p}{q}\cdot b}{a\pm\frac{p}{q}\cdot a}=\frac{d\pm\frac{r}{s}\cdot d}{c\pm\frac{r}{s}\cdot c}.$$

Whence, $a\pm\frac{p}{q}\cdot a : b\pm\frac{p}{q}\cdot b :: c\pm\frac{r}{s}\cdot c : d\pm\frac{r}{s}\cdot d.$

That is,

We may increase or decrease antecedent and consequent by like parts of each.

121. Let us have two proportions,

$$\left.\begin{array}{l} a : b :: c : d \\ f : g :: m : n \end{array}\right\} \text{ or, } \frac{b}{a}=\frac{d}{c};\ \frac{g}{f}=\frac{n}{m}.$$

Multiplying the equations, member by member,

$$\frac{bg}{af}=\frac{dn}{cm},$$

Whence, $af : bg :: cm : dn.$
That is,

We may multiply proportions together, term by term.

122. From $a : b :: c : d,$
we have $bc=ad$; adding ab to both members,

$$bc+ab=ad+ab.$$

Whence, $b(a+c)=a(b+d).$

And, from a previous principle,

$$a : b :: a+c : b+d.$$

Hence,

Antecedent is to its consequent, as the sum of the antecedents is to the sum of the consequents.

123. Where several proportions are written together, thus:

$$a : b :: c : d :: f : g :: \text{, etc.,}$$

it is called a *continued* proportion.

It is sometimes written thus,

$$a : c : f :: b : d : g.$$

The principles already explained may be readily extended to continued proportions.

124. The *mean proportional* between two quantities may be readily found: thus, let it be required to find the mean between a and b.

We have,

$$a : x :: x : b,$$

$$x^2 = ab,$$

$$x = \sqrt{ab}.$$

That is,

The mean of any two quantities may be found by multiplying the quantities together and extracting the square root of the product.

125. When one of two variables is expressed in terms of the reciprocal of the other, as,

$$x = \frac{1}{y}; \text{ or } x = \frac{m}{y},$$

the quantities are said to be *reciprocally proportional.* It is manifest that one will increase as the other diminishes, and they are thus said to *vary inversely.*

From this expression, we have,

$$xy = 1, \text{ or } xy = m.$$

Hence, the product of two such quantities is always constant.

126. Geometrical Progression.

A Geometrical Progression is a series, any term of which may be derived from the one preceding, by multiplying it by a constant quantity called the *ratio*. It is a continued proportion.

The progression will be *increasing* when the ratio is greater than unity: thus,

$$2,\ 4,\ 8,\ 16,\ 32,\ \text{etc.},$$

is an increasing progression, whose ratio is 2.

The progression will be *decreasing* when the ratio is less than unity; thus,

$$32,\ 16,\ 8,\ 4,\ 2,\ 1,\ \tfrac{1}{2},\ \text{etc.},$$

is a decreasing progression, in which $\frac{1}{2}$ is the ratio.

127. Formula for Last Term.

Let us assume the geometrical progression,

$$a : b : c : d : e : f : \text{etc.},$$

in which r is the ratio.

From the definition we shall have,

$$b=ar,\quad c=br=ar^2,\quad d=cr=ar^3,\ \text{etc.}$$

Whence we see that the exponent of r, being unity in the expression for b, goes on increasing by unity for c, d, etc.; so that for the term which has n terms before it, calling it l, we shall have,

$$l=ar^{n-1}.$$

Hence, we may say that,

Any term of a geometrical progression may be found by multiplying the first term by the ratio raised to a power whose exponent is equal to the number of preceding terms.

128. Formula for the Sum.

To find a formula for computing the sum of any number of terms of such a series, let us take a progression of a definite number of terms,

$$a : b : c : d : \text{- - - - -} : j : k : l,$$

the ratio being r.

Replacing each term after the first by its value in terms of the first term and the ratio, and representing the sum of n terms by s, we shall have,

$$s=a+ar+ar^2+ \text{ - - - } ar^{n-2}+ar^{n-1}.$$

Now, multiplying both members of this by r,

$$sr=ar+ar^2+ar^3+ \text{ - - - } ar^{n-1}+ar^{n}.$$

Subtracting the first of these equations from the second, we shall have,

$$sr-s=ar^n-a;$$

whence, $s=\dfrac{ar^n-a}{r-1}$; or, since ar^n is equal to lr,

$$s=\frac{lr-a}{r-1},$$

an expression for the sum of any given number of terms.

We have thus the two formulas,

$$l=ar^{n-1} \text{ and}$$

$$s=\frac{lr-a}{r-1},$$ in which there are five arbitrary quantities. If we know all but one which enter either of them, we can substitute the given values in such formula, and at once deduce the remaining one; thus, if $a=2$, $r=2$, and $n=5$, we shall have from the first,

$$l=2\times 2^4=32.$$

If $s=15$, $a=1$, and $r=2$, from the second formula,

$$15=\frac{2l-1}{2-1}; \text{ whence, } l=8.$$

129. To find the Formula for any Element.

When any three of the five quantities, a, l, r, n, and s, which enter these two formulas, are given, we may combine the formulas, and eliminate one of the remaining quantities, and thus find the fifth quantity. When a combination of the formulas is required, make the combination and deduce a general expression for the desired quantity, before substituting for the given quantities.

For example, let $s=62$, $r=2$ and $n=5$. Combining the formulas, eliminating l, we have,

$$a=\frac{(r-1)s}{r^n-1};$$

whence, substituting the given values,

$$a=\frac{(2-1)62}{2^5-1}=2.$$

Substituting $a=2$, $r=2$ and $n=5$ in the first formula, we have, $l=32$.

When n is to be found, the resulting formulas will require the use of logarithms, which are yet to be explained.

Examples.

1. Given $a=1$, $r=2$, $n=7$, to find l and s. *Ans.* $l=64$, $s=127$.

2. Given $a=4$, $r=3$, $n=10$, to find l and s.
Ans. $l=78732$, $s=118096$.

3. Given $a=9$, $r=\frac{7}{4}$, $n=7$, to find l and s.
Ans. $l=258\frac{1}{2}$, $s=591\frac{7}{40}$

4. Given $a=6\frac{1}{4}$, $r=\frac{1}{2}$, $n=8$, to find l and s.
Ans. $l=106\frac{4}{5}$, $s=307\frac{4}{5}$.

5. Given $a=8$, $r=\frac{1}{2}$, $n=15$, to find l and s.
Ans. $l=\frac{1}{2048}$, $s=15+$.

6. Given $a=\frac{5}{6}$, $r=\frac{2}{3}$, $n=11$, to find l and s.
Ans. $l=\frac{2560}{177147}$, $s=8\frac{452}{756}$.

130. To insert Geometrical Means.

Let it now be required to insert a given number of geometrical means between any two numbers, as a and b. These two quantities, together with the means, will form a geometrical progression of two more terms than the number of the means. If there are to be m means there will be $m+2$ terms. Now, from the formula

$$l=ar^{n-1},$$

we have,

$$r=\sqrt[n-1]{\frac{l}{a}}.$$

This is the value of the ratio when there are n terms, and a and l are the extremes. But in the case in hand, a and b are the extremes and $m+2$ the number of terms. Substituting these in the above expression for r, we shall have,

$$r=\sqrt[m+1]{\frac{b}{a}}. \quad - \quad - \quad (1)$$

This being the new ratio, we may now find the means successively from the first term a; thus,

$$a : a\sqrt[m+1]{\frac{b}{a}} : a\sqrt[m-1]{\frac{b^2}{a^2}} : a\sqrt[m+1]{\frac{b^3}{a^3}} : \cdot \cdot \cdot \cdot : b.$$

For example, let it be required to insert 3 geometrical means between 2 and 32.

In this case $m=3$, $a=2$, and $b=32$.

These in (1) give, $r=\sqrt[4]{\frac{32}{2}}=\sqrt[4]{16}=2.$

Whence the means will be 4, 8, and 16, and we shall have the progression,

$$2 : 4 : 8 : 16 : 32.$$

EXAMPLES.

1. Insert 5 geometrical means between 3 and 192.
Ans. 3 : 6 : 12 : 24 : 48 : 96 : 192.

2. Insert 4 geometrical means between 5 and 1215.
Ans. 5 : 15 : 45 : 135 : 405 : 1215.

131. The Sum of an Infinite Progression.

Let us now have a decreasing progression of an infinite number of terms. The ratio will be less than unity. The formula,

$$s=\frac{ar^n-a}{r-1}, \text{ may be written,}$$

$$s=\frac{ar^n}{r-1}-\frac{a}{r-1}.$$

But, since r is less than 1, r^2 will be less than r, r^3 still less, and so the results will go on diminishing as the power is increased; and since n is infinite, r^n is zero; whence it follows, that the first term of the second member of the above expression is zero, and we have

$$s=-\frac{a}{r-1}.$$

But, since $r<1$, we may write

$$s=\frac{a}{1-r}.$$

That is to say,

The sum of the terms of a decreasing geometrical progression of an infinite number of terms, is equal to the first term, divided by unity minus the ratio.

What is the sum of the following?

1. $1 : \frac{1}{2} : \frac{1}{4}$ - - - - - to infinity. *Ans.* 2.
2. $1 : \frac{1}{3} : \frac{1}{9}$ - - - - - to infinity. *Ans.* $\frac{3}{2}$.
3. $4 : 2 : 1$ - - - - - to infinity. *Ans.* 8.

4. $\frac{1}{5} : \frac{1}{25} : \frac{1}{125}$ - - - to infinity. *Ans.* $\frac{1}{4}$.

5. $1 : \dfrac{1}{a} : \dfrac{1}{a^2}$ - - - to infinity. *Ans.* $\dfrac{a}{a-1}$.

If the ratio of a geometrical progression is negative, the terms will be alternately positive and negative. In substituting in the formulas, be careful to give the ratio its proper sign.

6. Given, $a=-2$, $r=-3 n=5$, to find l and s.

Ans. $l=-162$, $s=-121$.

SECTION X.

LOGARITHMS.

132. If the number 10 be raised to the second power, we shall have 100 for the result; thus,

$$10^2=100.$$

The exponent 2 is called the *logarithm* of 100. Let us write several exact powers of 10, thus,

$$(10)^{-2},\ (10)^{-1},\ 10^0,\ 10^1,\ 10^2,\ 10^3, \text{ etc.}$$

Or,

$$\frac{1}{(10)^2},\quad \frac{1}{10},\quad 1,\quad 10,\quad 100,\ 1000, \text{ etc.}$$

The exponents,

$$-2,\ -1,\ 0,\ 1,\ 2,\ 3, \text{ etc.},$$

in the upper row, are the respective logarithms of

$$.01,\ .1,\ 1,\ 10,\ 100,\ 1000, \text{ etc.},$$

in the lower row.

If we should take any number which is not an exact power of 10, the exponent would not be a whole number, but would be made up of an entire part and a fraction; thus, the number 25 is greater than the first power of 10, and less than the second power; whence 10 must be raised to a power greater than 1, and less than 2, to produce 25. Assuming that the fraction to be added to 1 to give the proper exponent is .397940, we should have,

$$(10)^{1.397940}=25.$$

Here 1.397940 is the logarithm of 25.

It is obvious that any number except unity may be used instead of 10, as the fixed number, whose several powers are to be taken.

We may, then, say that,

The logarithm of a number is the exponent of the power to which it is necessary to raise a fixed number, called the base, in order to produce the given number.

The base of the *Common System*, as it is generally called, is 10.

The base of the *Napieran System*, so named from Baron Napier, to whom the invention of logarithms is due, is 2.71828182....

The entire part of a logarithm is called the *characteristic;* the decimal part is called the *mantissa.*

133. Characteristic.

The characteristic of the logarithm of a number may always be written at once from the number itself. The law for writing the characteristic may be discovered from the following:

$$
\begin{array}{lcl}
(10)^4 = 1,0000 & \therefore & \log 10,000 = 4. \\
(10)^3 = 1,000 & \therefore & \log 1,000 = 3. \\
(10)^2 = 100 & \therefore & \log 100 = 2. \\
(10)^1 = 10 & \therefore & \log 10 = 1. \\
(10)^0 = 1 & \therefore & \log 1 = 0. \\
(10)^{-1} = .1 & \therefore & \log .1 = -1. \\
(10)^{-2} = .01 & \therefore & \log .01 = -2. \\
(10)^{-3} = .001 & \therefore & \log .001 = -3. \\
(10)^{-4} = .0001 & \therefore & \log .0001 = -4.
\end{array}
$$

It thus appears that the characteristic of all numbers greater than unity is *positive,* and that the characteristic of all numbers less than unity is *negative;* and that,

When the number is greater than unity, the characteristic is one less than the number of digits composing the number. If the number is partly decimal, only the entire part is to be counted.

When the number is less than unity, and is expressed as a decimal, the characteristic is negative and one greater than the number of 0's immediately following the decimal point.

The mantissa, or decimal part, must be found in a table computed for the purpose.

134. The following is the beginning of such a table, showing the logarithms from 1 to 100:

TABLE.—LOGARITHMS FROM 1 TO 100.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0 000000	26	1 414973	51	1 707570	76	1 880814
2	0 301030	27	1 431364	52	1 716003	77	1 886491
3	0 477121	28	1 447158	53	1 724276	78	1 892095
4	0 602060	29	1 462398	54	1 732394	79	1 897627
5	0 698970	30	1 477121	55	1 740363	80	1 903090
6	0 778151	31	1 491362	56	1 748188	81	1 908485
7	0 845098	32	1 505150	57	1 755875	82	1 913814
8	0 903090	33	1 518514	58	1 763428	83	1 919078
9	0 954243	34	1 531479	59	1 770852	84	1 924279
10	1 000000	35	1 544068	60	1 778151	85	1 929419
11	1 041393	36	1 556303	61	1 785330	86	1 934498
12	1 079181	37	1 568202	62	1 792392	87	1 939519
13	1 113943	38	1 579784	63	1 799341	88	1 944483
14	1 146128	39	1 591065	64	1 806180	89	1 949390
15	1 176091	40	1 602060	65	1 812913	90	1 954243
16	1 204120	41	1 612784	66	1 819544	91	1 959041
17	1 230449	42	1 623249	67	1 826075	92	1 963788
18	1 255273	43	1 633468	68	1 832509	93	1 968483
19	1 278754	44	1 643453	69	1 838849	94	1 973128
20	1 301030	45	1 653213	70	1 845098	95	1 977724
21	1 322219	46	1 662758	71	1 851258	96	1 982271
22	1 342423	47	1 672098	72	1 857333	97	1 986772
23	1 361728	48	1 681241	73	1 863323	98	1 991226
24	1 380211	49	1 690196	74	1 869232	99	1 995635
25	1 397940	50	1 698970	75	1 875061	100	2 000000

135. General Principles of Logarithms.

The principles already established, Art. 50, with regard to exponents in general, govern the use of logarithms, they being but a class of exponents.

Let us repeat the principles referred to, however, in this connection, since they form the basis of all logarithmic computations.

Let m and n be any two numbers whose logarithms are x and y. Then,

$$(10)^x = m \quad - \quad - \quad (1)$$

$$(10)^y = n \quad - \quad - \quad (2).$$

Multiplying member by member, we have,

$$(10)^{x+y} = mn,$$

whence, $$x+y = \log mn,$$

That is,

1. *The sum of the logarithms of two numbers is the logarithm of their product.*

Again, from (1) and (2) we have,

$$\frac{(10)^x}{(10)^y}=\frac{m}{n}; \text{ or,}$$

$$(10)^{x-y}=\frac{m}{n}; \text{ whence,}$$

$$x-y=\log\frac{m}{n}.$$

That is,

2. *The difference of the logarithms of two numbers is the logarithm of their quotient.*

We have again, from (1),

$$\left((10)^x\right)^p=m^p \therefore$$
$$10^{px}=m^p \therefore$$
$$px=\log m^p.$$

That is,

3. *The product of the logarithm of a number by* p *is the logarithm of the* p *power of the number.*

If we extract the r root of both members of (1), we have,

$$(10)^{\frac{x}{r}}=\sqrt[r]{m} \therefore$$
$$\frac{x}{r}=\log\sqrt[r]{m}.$$

That is,

4. *The quotient of the logarithm of a number by* r, *is the logarithm of the* r *root of the number.*

These four truths are called *the general principles of logarithms.*

Apply the foregoing principles in the transformation of the following expressions into corresponding logarithm equations:

1. $x=\frac{ab}{cd}$. *Ans.* $\log x=\log a+\log b-\log c-\log d$.

2. $x=a^m b^n c^p$. *Ans.* $\log x=m\log a+n\log b+p\log c$.

3. $x=\frac{a^m b^{-n}}{c^p d^q}$. *Ans.* $\log x=m\log a-n\log b-p\log c-q\log d$.

4. $x=a^{\frac{m}{n}}b^{-\frac{p}{q}}$. *Ans.* $\log x=\frac{m}{n}\log a-\frac{p}{q}\log b$.

5. $x=\sqrt[n]{a^m b^{-n} c^{\frac{p}{q}}}$ *Ans.* $\log x=\frac{m}{n}\log a-n\log b+\frac{p}{q}\log c.$

6. $x=\frac{a\sqrt[n]{c^m}}{b\sqrt{d}}.$ *Ans.* $\log x=\log a+\frac{m}{n}\log c-\log b-\frac{1}{2}\log d.$

7. $x=\frac{(a+b)^n c^m}{(c+d)\sqrt[4]{d^3}}.$

Ans. $\log x=n\log(a+b)+m\log c-\log(c+d)-\frac{3}{4}\log d.$

8. $x=\frac{1}{\sqrt[n]{a+b}}.$ *Ans.* $\log x=-\frac{1}{n}\log(a+b).$

9. $x=\sqrt[m]{a^2-x^2}.$ *Ans.* $\log x=\frac{1}{m}\log(a+x)+\frac{1}{n}\log(a-x).$

10. $x=(a^m)^n.$ *Ans.* $\log x=mn\log a.$

136. Solution of Equations.

By the aid of logarithms we are enabled to solve what would otherwise be difficult equations.

Solve the following:

1. $\begin{cases} 5^x=25 \\ x\log 5=\log 25. \end{cases}$ *Ans.* $x=\frac{\log 25}{\log 5}.$

2. $7^{\sqrt{x}}=17.$

Whence, $\sqrt{x}\log 7=\log 17,$

which gives, $\sqrt{x}=\frac{\log 17}{\log 7}.$ *Ans.* $x=\left(\frac{\log 17}{\log 7}\right)^2.$

3. $a^x=b.$ *Ans.* $x=\frac{\log b}{\log a}.$

4. $(a+b)^{\sqrt[m]{x}}=c.$ *Ans.* $x=\left(\frac{\log c}{\log(a+b)}\right)^m.$

5. $a^{\frac{x}{m}}=b.$ *Ans.* $x=m\left(\frac{\log b}{\log a}\right).$

137. Logarithms of Decimals.

When a number is composed of the same figures, no matter how its value may be made to change by moving the decimal point to the right or left, the mantissa will always remain the same; the only change being in the characteristic: thus, if the logarithm of 2975376 is 6.354189, we shall have as follows:

$$\log 2975.376 = 3.354189.$$
$$\log 29.75376 = 1.354189.$$
$$\log 2.975376 = 0.354189.$$
$$\log .02975376 = \bar{2}.354189.$$

The mantissa, it will be observed, remains the same.

To prove that this is true, let n be any number whatever, and p any other whole number, positive or negative.

Then, $n \times (10)^p$, will represent the product of n by any exact power of 10. We may write,

$$\log (n \times (10)^p) = \log n + \log 10^p = \log n + p.$$

Now, in the expression, $\log n + p$, since p is entire, it will be added to, or if negative subtracted from, the characteristic; so that the decimal part will remain intact. But to multiply a number by any entire power of 10, positive or negative, is but to remove the decimal point to the right or left; and hence it follows that the mantissa of the logarithm of a number expressed by any combination of figures will remain the same, whether any part of it is decimal or not.

This principle enables us to disregard the decimal point entirely in finding the mantissa of the logarithms of numbers.

138. General Properties.

When the base of a system of logarithms is greater than unity, the logarithm of ∞ is $+\infty$, and the logarithm of 0 is $-\infty$.

For assuming the exponential equation,

$$10^x = n,$$

n being any number and x its logarithm, we see that as n increases x must increase, and when n is infinite x must be infinite also.

As n is made less and less, x must be diminished, until when n is 1, x is 0. If, now, n is still diminished, x becomes negative, and when n is 0, x is again ∞, and becoming so from the negative side, is still regarded as negative. Thus, the logarithms of positive numbers alone require for their logarithms all numbers from $-\infty$ to $+\infty$. There cannot, then, be any logarithms of negative quantities.

If the base is less than unity, the same thing will hold; except that the logarithms of numbers greater than unity will then be negative, and those of numbers less than unity will be positive.

It will be seen that the logarithm of unity in any system is zero.

139. Modulus.

Let us assume what is called the *Logarithmic Series,* leaving its deduction for a more advanced course. It is,

$$\log(1+y)=m(y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+ \text{ - - - - - etc.}),$$

in which we have the logarithm of a number, $(1+y)$, developed into a series, in terms of a number, y, less by unity than the number itself.

It will be observed that the second member is composed of two factors, m and the series within the brackets. The factor m depends for its value entirely upon the base of the system, and, for the same system, is constant. It is called the *modulus* of the system. The *modulus,* then, of any system of logarithms is the constant factor, which is common to all logarithms of that system.

The modulus of the Napieran system is 1; the modulus of the common system is 0.43429448.

140. Indeterminate and Identical Equations.

An *Indeterminate* quantity is one which has no fixed value, but admits of an infinite number of values in succession. *Indeterminate* and *arbitrary* quantities do not differ in nature; but it is usual to call such quantities, when represented by the first letters of the alphabet, arbitrary; and when represented by the final letters, indeterminate.

An *Indeterminate Equation* is one which contains at least two quantities which can only be assumed in relation to each other; thus,

$$5x-4y=9,$$

is such an equation; x and y admitting of any number of sets of values, but neither of them admitting of any specific value independently of the other. Such quantities are called indeterminate, or, perhaps more correctly, *variables.*

An *Identical Equation* is one which is true for all possible values of the indeterminate, or arbitrary quantities, which enter it. For example,

$$ax+by=ax+by,$$
$$(a+x)^2=a^2+2ax+x^2,$$
$$5x+7x^2+4=5x+7x^2+4,$$

are manifestly such equations.

An identical equation differs materially from the ordinary or determinate equation, which admits of but a specific number of values; and also from the indeterminate equation, which is satisfied for any number of sets of relative values of the variables which enter it. In an identical equation, the co-efficients of the indeterminate quantities (x, y, z, etc.), are no longer arbitrary, but have resultant values, and so are themselves determinate.

141. Indeterminate Co-efficients.

Any identical equation containing but one indeterminate quantity can be reduced to the form of

$$a+bx+cx^2+dx^3+\text{etc.}=0 \quad - \quad - \quad (1),$$

an equation in which a, b, c, d, etc., are called *Indeterminate Co-efficients,* since they are co-efficients of the indeterminate quantity.

Now, since this equation is true whatever value x may have, it will be true when $x=0$. Making this hypothesis in the equation, we shall have,

$$a=0.$$

This value of a in (1) gives,

$$bx+cx^2+dx^3+\text{etc.}=0.$$

Factoring,

$$x(b+cx+dx^2+\text{etc.})=0.$$

This may be satisfied for $x=0$, or,

$$b+cx+dx^2+\text{etc.}=0 \quad - \quad - \quad (2)$$

Now, x, in this equation, must admit of the same value as in (1); hence, this must also be an identical equation, and is true for $x=0$.

This value of x in (2) gives,

$$b=0;$$

and in the same way we may prove,

$$c=0,$$
$$d=0;$$

and so for all the co-efficients of x.

Hence, we may say that,

In an identical equation of one indeterminate quantity, the co-efficients of that quantity are severally equal to zero.

142. Principle of Indeterminate Co-efficients.

Let us now have the identical equation,

$$a+bx+cx^2+\text{etc.}=a'+b'x+c'x^2+\text{etc.}$$

Transposing and factoring,

$$a-a'+(b-b')x+(c-c')x^2+\text{etc.}=0.$$

Now, from the principle just deduced,

$$\begin{array}{ccc} a-a'=0 & \therefore & a=a' \\ b-b'=0 & \therefore & b-b' \\ c-c'=0 & \therefore & c=c' \\ \text{etc.} & & \text{etc.} \end{array}$$

Whence, we may say that,

In an identical equation of one indeterminate quantity, the co-efficients of the like powers of that quantity in the two members are severally equal to each other.

This principle may be readily extended to identical equations containing any number of indeterminate quantities. It is called *the Principle of Indeterminate Co-efficients.*

143. Development of Expressions.

Let us now apply the principle of Indeterminate Co-efficients to the development of algebraic expressions into series.

Take the fraction, $\frac{1+2x}{1-x-x^2}$, and placing it equal to a development of the proposed form: we have,

$$\frac{1+2x}{1-x-x^2}=a+bx+cx^2+dx^3+\text{ etc.} \quad - \quad (1).$$

Since this development must be true for all values of x, this must be an identical equation.

Clearing it of fractions, using the vertical bar for convenience, we have,

$$\begin{array}{r|r|r|l} 1+2x=a+b & x+c & x^2+d & x^3+\text{ etc.} \\ -a & -b & -c & +\text{ etc.} \\ & -a & -b & +\text{ etc.} \end{array}$$

Whence, by the principle of Indeterminate Co-efficients, we shall have,

$$\begin{array}{lll} 1=a & \therefore & a=1 \\ 2=b-a & \therefore & b=3 \\ 0=c-b-a & \therefore & c=4 \\ 0=d-c-b & \therefore & d=5 \\ \text{etc.} & & \text{etc.} \end{array}$$

These values, substituted for a, b, c, d, etc., in (1) give,

$$\frac{1+2x}{1-x-x^2}=1+3x+4x^2+5x^3+\text{etc.},$$

the desired development.

Examples.

1. Develop $\dfrac{1+2x}{1-3x}$ into a series. *Ans.* $1+5x+15x^2+45x^3+\text{etc.}$

2. Develop $\dfrac{1-x}{1+x+x^2}$ into a series.

Ans. $1-2x+x^2+x^3-2x^4+\text{etc.}$

3. Develop $\dfrac{a}{a+x}$ into a series. *Ans.* $1-\dfrac{x}{a}+\dfrac{x^2}{a^2}-\dfrac{x^3}{a^3}+\text{etc.}$

4. Develop $\dfrac{1-x}{1-2x-3x^2}$ into a series.

Ans. $1+x+5x^2+13x^3+\text{etc.}$

5. Develop $\dfrac{1}{2x+x^2}$ into a series.

In this case we have,

$$\frac{1}{2x+x^2}=a+bx+cx^2+\text{etc.}$$

$$1=2a\left|\begin{matrix}x\\ \end{matrix}\right.\begin{matrix}+2b\\ +\cdot a\end{matrix}\left|\begin{matrix}x^2\\ \end{matrix}\right.\begin{matrix}+2c\\ +\ b\end{matrix}\left|\begin{matrix}x^3\\ \end{matrix}\right.\begin{matrix}+\text{etc}\\ +\text{etc.}\end{matrix}$$

Whence,

$$\begin{array}{c} 1=0 \\ 0=2a \\ \text{etc.} \end{array}$$

Here we encounter an absurdity in the result, $1=0$.

This shows that the development cannot be made as proposed.

Then, let us factor the expression thus, $\dfrac{1}{x}\times\dfrac{1}{2+x}$: now let us develop $\dfrac{1}{2+x}$. We have,

$$\frac{1}{2+x}=a+bx+cx+\text{ etc.}$$

Proceeding as usual, we have,

$$\begin{array}{l} 1=2a+2b\left|x+2c\right|x+\text{etc.} \\ \qquad\quad\ \ a\ \ \qquad b\ \ +\text{etc.} \end{array}$$

$$\begin{array}{ll} 1=2a. & \therefore\ a=\ \frac{1}{2}. \\ 0=2b+a. & \therefore\ b=-\frac{1}{4}. \\ 0=2c+b. & \therefore\ c=\ \frac{1}{8}. \\ \quad\text{etc.,} & \quad\text{etc.} \end{array}$$

And thus we have,

$$\frac{1}{2+x}=\frac{1}{2}-\frac{1}{4}x+\frac{1}{8}x^2+\text{etc.}$$

Now, multiplying both members by $\frac{1}{x}$, we have,

$$\frac{1}{2x+x^2}=\frac{1}{2x}-\frac{1}{4}+\frac{1}{8}x+\text{etc};$$

which is the true development of the original expression.

Here the first term $\frac{1}{2x}$ is the same as $\frac{x^{-1}}{2}$; so that the absurdity above arose from not starting our development with a low enough power of x.

So, in general, when such an absurdity develops itself, it will be found due to a like cause.

143. Partial Fractions.

The Principle of Indeterminate Co-efficients affords an easy method of resolving a fraction into its partial fractions.

To do this, place the given fraction equal to the sum of as many partial fractions as the denominator of the fraction has factors, the several numerators of such assumed partial fractions being p, q, r, etc.—quantities to be determined—and the denominators being the several separate factors of the denominator.

For example, let it be required to find the partial of $\frac{2a}{a^2-b^2}$. We have,

$$\frac{2a}{a^2-x^2}=\frac{p}{a+x}+\frac{q}{a-x},\ \text{an identical equation.}$$

Clearing of fractions,

$$2a=ap-px+aq+qx.$$

Whence,

$$2a=\ ap+aq;$$
$$0=-p+q.$$

Combining these two equations and eliminating q, we have,

$$p=1.$$

This value of p gives,

$$q=1.$$

Therefore,

$$\frac{2a}{a^2-x^2}=\frac{1}{a+x}+\frac{1}{a-x}.$$

Examples.

1. Resolve $\frac{3x-3}{x^2-9}$ into partial fractions. *Ans.* $\frac{1}{x-3}+\frac{2}{x+3}$.

2. Resolve $\frac{x^2+4x-11}{(x-1)(x+2)(x-3)}$ into partial fractions.

Ans. $\frac{1}{x-1}-\frac{1}{x+2}+\frac{1}{x-3}$.

3. Resolve $\frac{1-2x+x^2}{(1-x)^3}$ into partial fractions.

When there are equal factors in the denominator, as in this case, use all of the several powers; thus,

$$\frac{1-2x+x^2}{(1-x)^3}=\frac{p}{(1-x)^3}+\frac{q}{(1-x)^2}+\frac{r}{1-x}.$$

Ans. $\frac{1}{1-x}-\frac{1}{1-x}+\frac{1}{1-x}$.

4. Resolve $\frac{a+x}{(a+x)^2}$ into partial fractions. *Ans.* $\frac{2}{a+x}-\frac{1}{a+x}$.

5. Resolve $\frac{-16}{x^2+2x-15}$ into partial fractions.

Ans. $\frac{2}{x+5}-\frac{2}{x-3}$.

NOTE.

PROBLEM OF THE LIGHTS.

THE point maintained in the discussion of the problem of the lights, that $c=0$, when a is greater or less than b, is a legitimate hypothesis, and that the analytical result is to be interpreted in a strict and natural way, is so entirely at issue with all the late writers upon algebra, that a more detailed examination of the question seems to be demanded than could well be given it in the body of the text.

Let us, then, consider, first, the arguments upon which we are asked to abandon the analysis in favor of what is supposed to be common-sense.

It will be remembered that the roots of the equation derived from the conditions of the problem are, $x=\frac{c\sqrt{a}}{\sqrt{a}\pm\sqrt{b}}$. Now, when c is made equal to zero in these roots, a and b remaining unequal, we must have $x=0$, for the value of either root. The interpretation of this result, according to the general law, gives us the origin of distances as the single point of equal illumination. We then have the case of two lights, placed at the same point, differing, however, much in intensity, and are required, by the analysis, to conclude that such point is equally illuminated by the greater and the lesser light. This conclusion, we are told, is not true.

The first distinguished author, whose reasoning upon the point we shall examine, says, in his University Algebra, "It is obvious that when two lights, of unequal intensities, occupy the same place, there is no point in space equally illuminated by them ; not even the point in which they are both situated."

To show how the result, $x=0$, in this case fails, this author carries us back to the original equation, $\frac{a}{x^2}=\frac{b}{(c-x)^2}$; which, we agree with him, "truly represents the conditions of the problem." He says, "If we put $c=0$, the result is $\frac{a}{x^2}=\frac{b}{x^2}$; an equation which cannot be satisfied *by any value of x whatever*, while $a>b$ or $a<b$. For, by substituting any value for x, we shall always obtain two unequal fractions. If $x=0$, the two members are two *unequal infinities*." And so he concludes, that, under this hypothesis, "the problem *fails* altogether, and is *impossible*." But can two infinite distances be unequal? It is to be presumed that no one will dispute that an infinite length is a distance than which nothing can be conceived to be greater. But to say that two distances are unequal in length, is incontestably to limit the less, and so to reduce it at once to the finite ; hence, one of this author's infinities must be finite—a contradiction in terms. The argument failing, the analytical deduction, $x=0$, still demands our confidence.

But, if possible, a more surprising turn, to avoid the difficulty, is made by another eminent authority. It is the more marked, too, in this case, from the fact that there has been a deliberate departure in the last edition of this author's

University Algebra, as well as in his more elaborate work—a work which has long held its place as a standard authority. In the previous editions the result of the analysis upon this point was accepted as true, but without any attempt at an explanation of the difficulty. We are now, however, told that,

"The hypothesis of $c=0$ places the two lights at the same point, in which case they form one and the same light, whose intensity is equal to the sum of their intensities taken separately. The conditions of the problem involve the necessity of *two lights*, and the equation of the problem is found under this hypothesis. This equation ought not, therefore, to respond to the case of a single light. For, the interpretation of the result obtained from one equation, can only give the cases which fall under the hypothesis. The hypothesis of two lights, and the hypothesis of a single light, are not connected by any law which affords a common equation of condition. Hence, the results obtained on the supposition of $c=0$, do not belong to the problem."

Now this reasoning has a very plausible look; but it should seem that it will not bear investigation. In the first place, the two lights do not become one in the writer's sense, any more than two circles of unequal radii become one when they have a common centre. Again, if one of the lights be removed, the other will remain, and since there is nothing requiring the lights to act simultaneously, the real question in issue is to find the point or points which would be equally illuminated by either light, separately, and, if you please, at times, however remote from each other.

Again, there is a most unmistakable connection between the two lights—the very arbitrary quantity c in question. Suppose it should be required to find the locus of a point moving so that its distance from two fixed points shall be equal to a given line. Here are two points, but shall we be told that the distance between them may not be made zero, and so give us the circle—a particular case of the ellipse? or that, having the equation of a line passing through two points, the two points may not be made one without destroying the equation?

Again, this author tells us, in the works in question, that the discussion of a problem consists in making every possible supposition upon the arbitrary quantities which enter it, and interpreting the results. How does this agree with his assertion in this case, that the "results obtained by making $c=0$ do not belong to the problem"? Are we to understand that he considers it impossible to make this hypothesis in an equation containing c? Indeed the doctrine here put forth would make short work with much of the higher mathematics, and the whole doctrine of limits.

Nor would the difficulty in point be at all obviated, if one were to admit the whole of this writer's reasoning. The equation must still respond, so long as c is not absolutely zero. Well, now suppose the lights to be, the one very great and the other very small, and that c is an infinitesimal. We shall have the lesser light occupying the second or third consecutive point from the greater. We must have one point of equal illumination between them; but will common sense with regard to lights, which seems to frighten our author into deserting the analysis, be in any manner better saved? Will not the feebler one be hopelessly outshone by the more splendid luminary?

But a third, and the latest writer upon this point, pronounces the results of the analysis under the hypothesis in question, to be "evidently absurd." He says, "In discussing this problem, some have committed the error of considering that, since for $c=0$ and a and b unequal, $x=c\dfrac{\sqrt{a}}{\sqrt{a}\pm\sqrt{b}}=0$, therefore, there is a point of equal illumination at the point where the lights are situated! This is evidently absurd, since the hypothesis is that the lights are of *unequal* intensity. The error consists in not perceiving that the hypothesis, $c=0$, excludes the hypothesis, a and b unequal. That the hypotheses, $a>$ or $<b$, are excluded by the hypothesis, $c=0$, and that there is a point of equal illumination, is self-evident."

It is to be presumed that the learned author uses the word "self-evident" in an unusual sense, since the question he disposes of so summarily in the end, had already cost him quite an argument, to say nothing of having been a standing puzzle for many years. This conclusion is further sustained by the fact that he afterwards says, "Perhaps the student may think that these conditions are no more inconsistent than those in I. 3 above, viz., c finite and $a=b$, and a point of equal illumination," etc.

He goes on further to prove his "self-evident" proposition as follows:

"Also, that $a>$ or $<b$ is inconsistent with $c=0$ and $\dfrac{a}{x^2}=\dfrac{b}{(c-x)^2}$ (that is, that there is a point of equal illumination), appears from the fact that $c=0$ renders the latter $\dfrac{a}{x^2}=\dfrac{b}{x^2}$, whence $a=b$."

This writer seems to be unconscious of the fact that he has here stricken out zero as a factor from both members of the equation; thus, $ax^2=bx^2$, $a\cdot 0=b\cdot 0$ $\therefore$ $a=b$; by which process it is easy enough to prove $2=4$; a result which follows at once, if we assume $a=2$, and $b=4$, in the above equation. It should seem that the anxiety to escape a point which is thought to militate against common sense, has led to some hasty writing.

And now let us look at the question upon its own merits. It should seem that the chief difficulty arises from a most unwarrantable mixing up of physical phenomena and analytical exactness. Analysis, *per se*, knows nothing whatever of the physical world; nor does it in the slightest degree regard any language but its own. It takes our conditions and renders its verdict, leaving us to look out for ourselves and our physical applications.

The confusion in the minds of those authors who desert the analysis in this question, and betake themselves to what they mistake for common sense, arises from not sufficiently regarding the assumed law of physics under which the equation in this problem is established. These are purely theoretical lights, and the law is absolute, that their intensity shall vary inversely as the square of the distance. When we pass within the distance unity, their intensities increase with wondrous rapidity, until at zero—the limit of approach—we must have $\dfrac{a}{0}$, and $\dfrac{b}{0}$, for their respective intensities: that is to say, the intensity of either light at this point *is absolutely infinite*; and thus, unless we may indeed have "two unequal infinities," when the lights occupy the same point, that point must be equally illuminated, however much the lights may differ in intensities at the distance unity.

This may not, probably does not, accord with physical facts, which deal with tallow candles, or at best calcium lights; but let us not charge the failure upon the analysis, without first examining our assumed law of optics. That law should undoubtedly read, "the intensity of a light at any [*sensible*] distance is equal to its intensity at the distance unity, divided by the square of that distance." We should then be saved from the hypothesis in question, and from the physical anomaly.

But in further support of the analysis, and to show how wondrously it responds to every hypothesis, let us rid the question entirely of physical phenomena, and present it as an abstract question.

Let y represent the general ordinate of a curve which shall be equal to a when the abscissa is made equal to unity, and which shall vary inversely as the square of the abscissa. Then,

$$y=\frac{a}{x^2} \quad . \quad . \quad . \quad (1)$$

will be the equation of such curve. This equation may be readily constructed, and we shall have a curve with two branches, BC and DE as shown in Fig. 1.

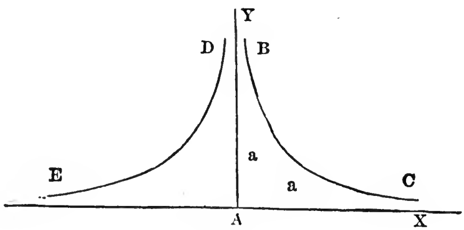


FIG. 1.

By making x and y zero in succession in equation (1), it appears at once that the axes are asymptotes to either branch, and thus that y is infinite when x is zero, and zero when x is infinite.

Let us now have another such curve whose equation is,

$$y=\frac{b}{(c-x)^2} \quad . \quad . \quad . \quad (2).$$

This curve gives y equal to b, when $c-x$ is equal to unity; and the ordinate whose abscissa is equal to c, is an asymptote to the curve.

Now, let it be required to find the common points of these two curves; that is, the points whose ordinates (they corresponding to the intensities in the problem) are equal. From (1) and (2), we must have,

$$\frac{a}{x^2}=\frac{b}{(c-x)^2};$$

the old equation, but with no question of light in it. The values of x may now be found as before, and the points constructed. The curves would, under the first hypothesis in the problem, take the general form shown in Fig. 2.

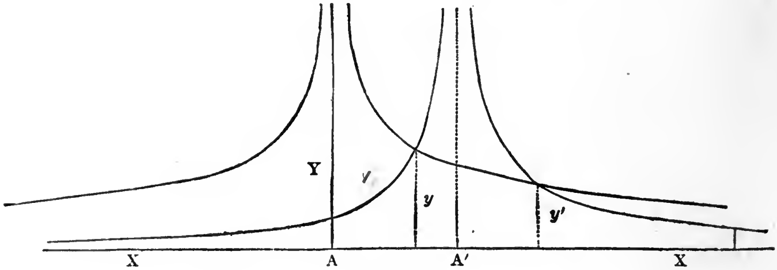


FIG. 2.

These curves will show to the eye the results of all the hypotheses which may be made in the problem. We shall make only the one in dispute, namely, $c=0$, and $a>$ or $<b$. The hypothesis $c=0$ makes the asymptotical ordinate coincide with the axis of Y, and the curves will assume the position shown in Fig. 3.

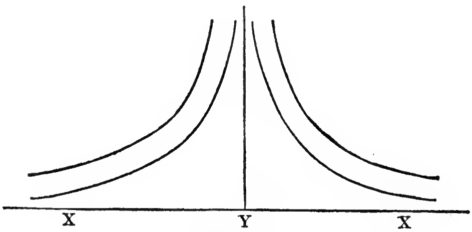


FIG. 3.

It will be seen that both curves approach the axis of Y, and that neither can ever reach it—the ordinate at the point zero being infinite—that is, resuming the question of lights, the intensities of both lights are *infinite* at the origin, and that point is equally illuminated. It will hardly be said that one of these curves reaches the axis before the other.

THE END.

APPENDIX.

ADDITIONAL EXERCISES.

Articles 28–30.

1. Simplify $21x+7y-5x-6y+10-16x-2$.

2. " $3x+4x^2y-5x+2x^2y+2x-5x^2y$.

3. " $2a^{\frac{1}{2}}-3b^{\frac{1}{3}}c+7a^{\frac{1}{2}}-9b^{\frac{1}{3}}c-9a^{\frac{1}{2}}$.

4. " $a^{\frac{1}{2}}b^m-a^{\frac{1}{3}}b^n+a^{\frac{1}{2}}b^m+5a^{\frac{1}{3}}b^n-2a^{\frac{1}{2}}b^m$.

5. " $3\sqrt{a}-4\sqrt{ab}+5\sqrt{a}-6\sqrt{ab}-7\sqrt{a}$.

6. " $2\sqrt{a+b}-3\sqrt[3]{a^2}-\frac{1}{2}\sqrt{a+b}+\frac{1}{3}\sqrt[3]{a^2}$.

7. " $7(a-b)^{\frac{1}{3}}+5(a+b)^{\frac{1}{2}}-(a-b)^{\frac{1}{3}}-7(a+b)^{\frac{1}{2}}$.

8. " $\dfrac{\sqrt{x+y}}{\sqrt{x}}+\frac{1}{2}\dfrac{\sqrt{x+y}}{\sqrt{x}}-\dfrac{\sqrt{xy}}{\sqrt{z}}-\frac{1}{3}\dfrac{\sqrt{xy}}{\sqrt{z}}-\frac{1}{5}\dfrac{\sqrt{x+y}}{\sqrt{x}}$.

9. " $3x+4y-(x+y-\frac{1}{2}x+\frac{1}{3}y)$.

10. " $4x^2y-3x^{\frac{1}{2}}y^3-\left(5x^2y+\frac{1}{2}x^{\frac{1}{2}}y^3-x^2y+\dfrac{x^2y}{2}\right)$.

11. " $\sqrt{a}-\sqrt[3]{b}+\dfrac{a}{b}-\left(\frac{1}{2}\sqrt{a}+\frac{1}{5}\sqrt[3]{b}+\frac{3}{5}\cdot\dfrac{a}{b}\right)$.

12. " $\dfrac{a-b}{c}+\dfrac{\sqrt{a-b}}{c}-\left(\dfrac{a-b}{2c}+\dfrac{\sqrt{a-b}}{\frac{1}{2}c}\right)$.

Articles 33–35.

1. $5xy\times 3x^2y^3$.

2. $4abc\times -7a^2b^3c^4$.

3. $\frac{1}{2}a^{\frac{1}{2}}b^2\times\frac{1}{3}a^2b^{\frac{1}{2}}$.

4. $2a^mb^m\times -5a^mb^n$.

5. $\frac{1}{3}a^mb^nc^p\times a^pb^nc^m$.

6. $a^{-1}b^{\frac{1}{2}}c^m\times a^2bc^m$.

7. $27a^{\frac{m}{n}}b^{\frac{n}{m}}c^{-m}\times -52a^{\frac{m}{n}}b^{-\frac{n}{m}}c^m$.

8. $5a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}} \times 17a^{\frac{1}{3}}b^{\frac{2}{5}}c^{\frac{3}{4}}$.

9. $a^{\frac{1}{2}} \times a^{\frac{1}{3}} \times a^{\frac{1}{4}} \times a^{\frac{1}{5}}$.

10. $x^{m+n-1}y^{m-n} \times x^{m-n+1}y^{m+n-1}$.

11. $\frac{1}{3}a^{\frac{p}{q}}\, b^{-m}c^{-\frac{1}{3}} \times -\frac{1}{4}a^{-\frac{p}{q}}b^{m}c$.

12. $5^{\frac{1}{2}}3^{\frac{1}{3}}2^{2} \times 3^{2}2^{\frac{1}{2}}5^{-1}$.

13. $\dfrac{ab}{cd} \times \dfrac{ab}{cd}, \qquad \dfrac{a^3b^2}{c} \times \dfrac{ab^3}{c^2}$.

14. $\dfrac{a^{\frac{1}{2}}b^{\frac{1}{3}}}{c} \times \dfrac{a^{-1}b^{-1}}{c^2}, \qquad \dfrac{5a^mb}{7ab^p} \times \dfrac{4a^mb^0}{3a^{-1}b^p}$.

Article 36.

1. $(a^2)^3,\ (a^3)^5,\ \left(a^{\frac{1}{2}}\right)^{\frac{1}{2}},\ (-a^m)^n,\ (a^{-m})^{-m}$.

2. $\left(\dfrac{2a^{\frac{1}{2}}b^{\frac{1}{2}}}{3d^{-2}}\right)^2,\ \left(\dfrac{5a^2b^{-m}c}{2x^{-1}y^3z^n}\right)^m,\ \left(\dfrac{a^{\frac{m}{n}}b^{\frac{n}{m}}c^{-\frac{1}{2}}}{\frac{1}{2}a^{\frac{1}{2}}b^{\frac{2}{3}}c^{-m}}\right)^{\frac{m}{n}}$.

Article 37.

1. $\sqrt{a^2b^4},\ \sqrt{a^4b^6},\ \sqrt[3]{a^3b^6},\ \sqrt[3]{x^3y^6z^3},\ \sqrt{25x^2y^4}$.

2. $\sqrt{\dfrac{16a^2b^4}{36x^4y^6}},\ \sqrt[3]{-\dfrac{8a^3b^6}{27c^3d^{-3}}},\ \sqrt[m]{\dfrac{a^{2m}}{b^{3m}}},\ \sqrt[5]{-\dfrac{a^5}{32b^5c^{10}}}$.

Article 38.

1. Simplify $\dfrac{22a^2b^4}{11ab^2},\quad \dfrac{6x^{\frac{1}{2}}y^2z^2}{2x^{\frac{1}{4}}yz^2},\quad \dfrac{a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}}}{a^{\frac{1}{4}}b^{\frac{1}{6}}c^{-\frac{1}{4}}}$.

2. " $\dfrac{6(a+b)^2}{2(a+b)},\quad \dfrac{(a-b)^{\frac{1}{2}}(a+b)^{\frac{1}{3}}}{(a-b)^{\frac{1}{4}}(a+b)^{\frac{1}{6}}},\quad \dfrac{(\sqrt{a-b})^{2m}(\sqrt{a+b})^{\frac{1}{2}}}{(\sqrt{a+b})^{\frac{1}{3}}(\sqrt{a-b})^{-m}}$.

Article 40.

Rid the following expressions of negative exponents :

1. $a^{-1},\ ab^{-1},\ a^{\frac{1}{2}}b^{-\frac{1}{2}},\ x^{-m}y,\ 7a^{-3}bc^{-n}$.

2. $\dfrac{a^{-1}b}{ab^{-1}}, \quad \dfrac{(a+b)^{-1}c}{(a-b)^{-m}d}, \quad \dfrac{(a+b)^{-1}c^{-\frac{1}{2}}d}{a(a-b)^{-2}d^{-1}}, \quad \dfrac{\left(\dfrac{a}{b}\right)^{-1}\left(\dfrac{c}{d}\right)}{\dfrac{1}{2}\left(\dfrac{x}{y}\right)^{-3}}.$

In the following expressions convert numerators into denominators, and the converse :

$$\frac{a^{-1}b^{-2}}{c^{-2}d^{-1}}, \quad \frac{(a+b)^{-2}(a-b)}{25abc}, \quad \frac{\left(\dfrac{a}{b}\right)\left(\dfrac{c}{d}\right)}{ab(a-b)^{-1}}, \quad \frac{-1}{5a^{\frac{1}{2}}b^{-1}c}.$$

Article 41.

Transform the following into equivalent expressions, using fractional exponents :

1. $\sqrt{a}, \quad \sqrt[3]{a^2}, \quad \sqrt{a+b}, \quad \sqrt{(a+b)(a-b)}, \quad \sqrt[3]{a(a-b)}.$

2. $\sqrt{\dfrac{5}{x}}, \quad \sqrt{\dfrac{3a}{a+b}}, \quad \sqrt[3]{\dfrac{a^2b^3}{10c}}, \quad \sqrt[3]{\dfrac{a(a-b)}{(c-d)^{\frac{1}{2}}}}, \quad \sqrt[n]{\dfrac{x\sqrt{y}}{\sqrt{x-y}}}.$

Article 42.

Transform the following into equivalent expressions, using the radical sign :

$$\frac{a^{\frac{1}{2}}}{b^{\frac{1}{3}}}, \quad \frac{(a-b)^{\frac{1}{2}}}{(c-d)^{\frac{1}{3}}}, \quad \frac{a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}}}{(d-c)^{\frac{1}{n}}}, \quad \left(\frac{a}{b}\right)^{\frac{1}{2}}, \quad \left(\frac{c}{d}\right)^{\frac{m}{n}}, \quad \frac{(5)^{\frac{1}{2}}(a+b)^p}{((c-d)^n)^{\frac{1}{2}}}.$$

Article 43.

Change the following into equivalent expressions, multiplying the indices first by 2, and then by 3, and then by m :

$$\sqrt{ab^2}, \quad \sqrt[3]{a^2(a-b)}, \quad \sqrt[m]{\frac{1}{x^n(x-y)^n}}, \quad \sqrt[n]{\frac{a^{\frac{m}{n}}}{a^{m-n}b^{-p}}}.$$

Article 44.

Transform the following into equivalent expressions having common indices :

1. $\sqrt{a}, \quad \sqrt[3]{b}, \quad \sqrt[5]{c}.$

2. $5\sqrt{a+b}, \quad a\sqrt[3]{5a^2b^3}, \quad 3b\sqrt{25-(a+b)^{-1}}.$

3. $\dfrac{1}{2}\sqrt{\dfrac{1}{a}}, \quad \dfrac{a}{b}\sqrt[3]{\dfrac{a}{b}}, \quad c\sqrt[4]{\dfrac{a^3}{b^5}}.$

4. $\sqrt[m]{a}, \quad \sqrt[n]{b^m}, \quad \sqrt[p]{c^n}.$

Article 45.

1. $\sqrt{a^3}\times\sqrt{a-b},\ \sqrt{a^3}\times\sqrt[3]{a-b},\ 7x\sqrt{a}\times 9y\sqrt[3]{a^2b^3}.$

2. $3\sqrt{-3x}\times 4\sqrt[3]{5y^2}\times 5\sqrt{2x^2y^3},\ 3a(a-b)^{\frac{1}{2}}\times a(5a^2b^3)^{\frac{1}{3}}.$

4. $\sqrt[n]{\frac{a}{b}}\times\sqrt[m]{\frac{a}{b}}\times-\sqrt[p]{\frac{a}{b}},\ \sqrt[m]{a^n}\times\sqrt[n]{a^m}.$

Article 46.

1. $\dfrac{\sqrt{5}}{\sqrt{10}},\ \dfrac{3\sqrt{a^3}}{5\sqrt{a^2}},\ \dfrac{5a^2\sqrt{\frac{1}{2}}}{-3\sqrt{\frac{1}{3}}},\ \dfrac{\frac{1}{2}\sqrt{a}}{\frac{1}{3}\sqrt{b}},\ \dfrac{\frac{b}{a}\sqrt{\frac{a}{b}}}{\frac{a}{b}\sqrt{\frac{b}{a}}}.$

2. $\dfrac{\sqrt{5}}{\sqrt[3]{5}},\ \dfrac{a\sqrt{b}}{\frac{c}{d}\sqrt[3]{b}},\ \dfrac{a\sqrt[m]{a}}{b\sqrt[m]{b}},\ \dfrac{-a^{\frac{1}{m}}b^{\frac{1}{m}}}{\sqrt{c}},\ \dfrac{(a-b)^{\frac{1}{n}}}{(a+b)^{\frac{1}{m}}}.$

Article 47.

Simplify the following:

1. $\sqrt[3]{x^3y-a^2x^3},\ \sqrt[5]{-32a^{10}b^6},\ \sqrt[3]{\frac{8a^3b}{27cd}},\ (m^2nx^5y^3)^{\frac{1}{2}}.$

2. $5a\sqrt{4a^2b(x-y)^4},\ 25abc\sqrt[3]{a^3b^3(a-b)^6},\ \sqrt[m]{a^m(a+b)^{-m}}.$

Article 49.

1. $\sqrt{36x^2y}+\sqrt{25y},\ \sqrt[3]{8x}+\sqrt[3]{xy^6}+\sqrt[3]{27x^4}.$

2. $\sqrt{48ab^2}+b\sqrt{75a},\ 7\sqrt[3]{2a}+5\sqrt[6]{4a^2}.$

3. $\sqrt[3]{8a^3b+16a^4}-\sqrt[3]{b^4+2ab^3},\ \sqrt{49a+245}-\sqrt{25a+125}.$

Article 79.

1. Solve, $\dfrac{x-2}{4}-\dfrac{3x}{2}+\dfrac{15x}{2}=37.$

2. " $\dfrac{x}{a}-\dfrac{a+b}{c}-\dfrac{3(x-1)}{5}=f.$

3. " $\dfrac{3x-2}{7}-\dfrac{3x+2}{11}=x-5.$

4. " $\dfrac{x-1}{5}=\dfrac{3x+2}{10}+4.$

5. " $x-\dfrac{3x-5}{13}+\dfrac{4x-2}{11}=x+1.$

6. " $\dfrac{3ax}{b}-\dfrac{2bx}{2}-5=\dfrac{a}{bc}.$